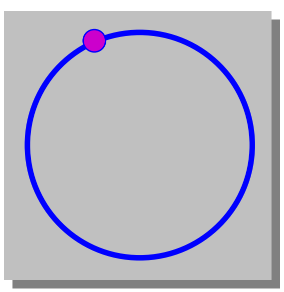
Differential Constraints

Beyond Points and Springs

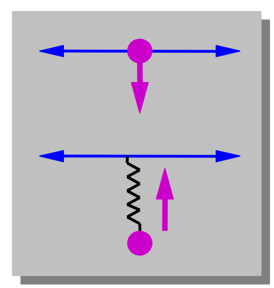
- You can make just about anything out of point masses and springs, *in principle*
- In practice, you can make anything you want as long as it's jello
- Constraints will buy us:
  - Rigid links instead of goopy springs
  - Ways to make interesting contraptions

A bead on a wire



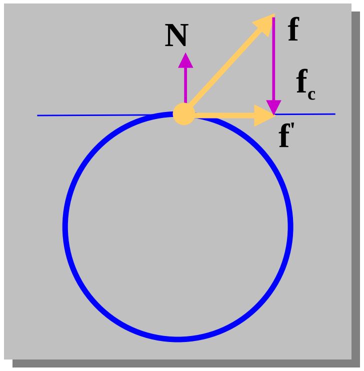
- Desired Behavior:
  - The bead can slide freely *along* the circle
  - It can never come off, however hard we pull
- Question:
  - How does the bead move under applied forces?

Penalty Constraints



- Why not use a spring to hold the bead on the wire?
- Problem:
  - Weak springs ⇒ goopy constraints
  - Strong springs ⇒ neptune express!
- A classic *stiff system*

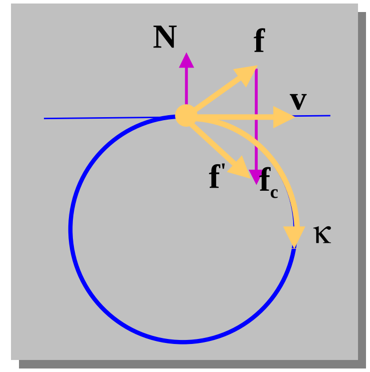
The basic trick (f = mv version)



- 1st order world.
- *Legal velocity:* tangent to circle (N·v = 0)
- *Project* applied force f onto tangent: $\mathbf{f}' = \mathbf{f} + \mathbf{f}_c$
- Added normal-direction force $\mathbf{f}_c$: *constraint force*
- No tug-of-war, no stiffness

$$\mathbf{f}_c = -\frac{\mathbf{f}\cdot\mathbf{N}}{\mathbf{N}\cdot\mathbf{N}}\mathbf{N} \qquad \mathbf{f}' = \mathbf{f} + \mathbf{f}_c$$

f = ma

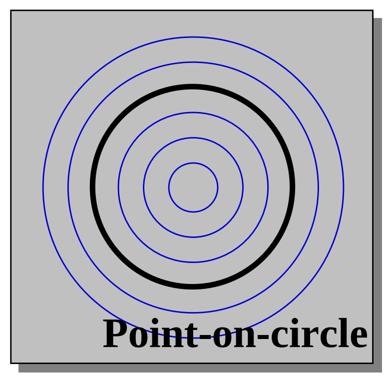


- Same idea, but…
- *Curvature* (κ) has to match.
- κ depends on *both* a and v:
  - the faster you're going, the faster you have to turn
- Calculate $\mathbf{f}_c$ to yield a legal *combination* of a and v
- Not as simple!

Now for the Algebra …

- Fortunately, there's a general recipe for calculating the constraint force
- First, a single constrained particle
- Then, generalize to constrained particle systems

Representing Constraints



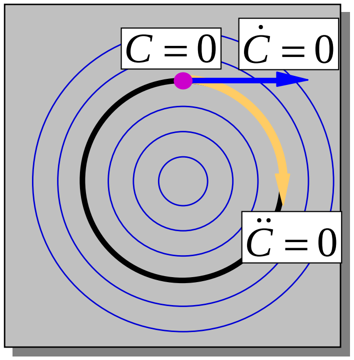
I. Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

~~*II. Parametric:*~~

~~$$\mathbf{x} = \mathbf{r}\left[\cos(\theta), \sin(\theta)\right]$$~~

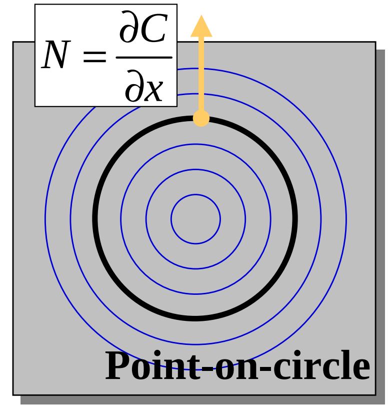
Maintaining Constraints Differentially



- **Start with legal position and velocity.**
- **Use constraint forces to ensure legal curvature.**

$C = 0$	legal position
$\dot{C} = 0$	legal velocity
$\ddot{C} = 0$	legal curvature

Constraint Gradient



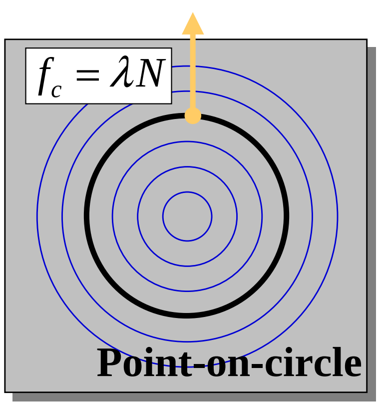
Implicit:

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

Differentiating C gives a normal vector.

This is the direction our constraint force will point in.

Constraint Forces



Constraint force: gradient vector times a scalar λ

Just one unknown to solve for

Assumption: constraint is passive—no energy gain or loss

Constraint Force Derivation

$$C(x(t))$$

$$f_c = \lambda N$$

$$\dot{C} = N \cdot \dot{x}$$

$$\ddot{x} = \frac{f + f_c}{m}$$

$$\ddot{C} = \frac{\partial}{\partial t}(N \cdot x)$$

$$= \dot{N} \cdot \dot{x} + N \cdot \ddot{x}$$

Set C̈ = 0, solve for λ:

$$\lambda = -m \frac{\dot{N} \cdot \dot{x}}{N \cdot N} - \frac{N \cdot f}{N \cdot N}$$

Constraint force is λN.

Notation: $N = \frac{\partial C}{\partial x}, \dot{N} = \frac{\partial^2 C}{\partial x \partial t}$

Example: Point-on-circle

$$C = |\mathbf{x}| - r$$

Write down the constraint equation.

$$\mathbf{N} = \frac{\partial \mathbf{C}}{\partial \mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

$$\dot{\mathbf{N}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{x} \partial t} = \frac{1}{|\mathbf{x}|}\left[\dot{\mathbf{x}} - \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{\mathbf{x}\cdot\mathbf{x}}\mathbf{x}\right]$$

Take the derivatives.

Substitute into generic template, simplify.

$$\lambda = -m\frac{\mathbf{N}\cdot\dot{\mathbf{x}}}{\mathbf{N}\cdot\mathbf{N}} - \frac{\mathbf{N}\cdot\mathbf{f}}{\mathbf{N}\cdot\mathbf{N}} = \frac{1}{|\mathbf{x}|}\left[m\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{\mathbf{x}\cdot\mathbf{x}} - m(\mathbf{x}\cdot\dot{\mathbf{x}}) - \mathbf{x}\cdot\mathbf{f}\right]$$

Drift and Feedback

- **In principle, clamping $\ddot{C}$ at zero is enough**
- **Two problems:**
  - **Constraints might not be met initially**
  - **Numerical errors can accumulate**
- **A feedback term handles both problems:**

$$C = -\alpha C - \beta\dot{C}, \text{ instead of}$$
$$\ddot{C} = 0$$

α and β are magic constants.

Tinkertoys

- **Now we know how to simulate a bead on a wire.**
- **Next: a constrained particle *system.***
  - **E.g. constrain particle/particle distance to make rigid links.**
- **Same idea, but…**

Constrained particle systems

- **Particle system: a point in state space**
- **Multiple constraints:**
  - **each is a function $C_i(x_1,x_2,\ldots)$**
  - ***Legal state:* $C_i = 0, \forall i$**
  - ***Simultaneous* projection**
  - **Constraint force: *linear combination* of constraint gradients**
- **Matrix equation**

Compact Particle System Notation

$$\ddot{\mathbf{q}} = \mathbf{W}\mathbf{Q}$$

q: *3n-long state vector.*

Q: *3n-long force vector.*

M: *3n* **x** *3n* **diagonal** ***mass matrix.***

W: M-inverse (element- wise reciprocal)

$$\ddot{\mathbf{q}} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$
$$\mathbf{Q} = [\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n]$$
$$\mathbf{W} = \begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix}$$
$$\mathbf{W} = \mathbf{M}^{-1}$$

Particle System Constraint Equations

Matrix equation for λ

$$[\mathbf{J}\mathbf{W}\mathbf{J}^T]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - [\mathbf{J}\mathbf{W}]\mathbf{Q}$$

Constrained Acceleration

$$\ddot{\mathbf{q}} = \mathbf{W}[\mathbf{Q} + \mathbf{J}^T\lambda]$$

Derivation: just like bead-on-wire.

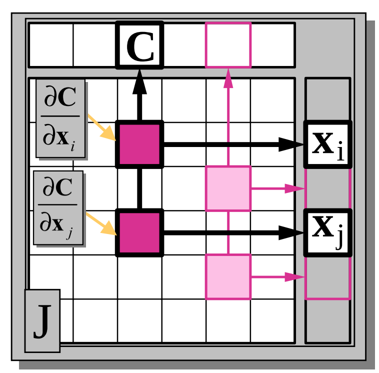
More Notation

$$\mathbf{C} = [C_1, C_2, \cdots, C_m]$$
$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_m]$$
$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$
$$\dot{\mathbf{J}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{q} \partial t}$$

How do you implement all this?

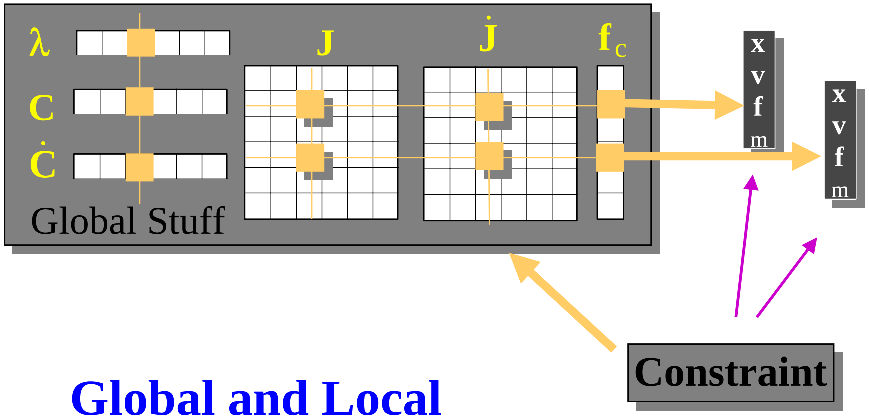
- **We have a global matrix equation**
- **We want to build models on the fly, just like masses and springs**
- **Approach:**
  - **Each constraint adds its own piece to the equation**

Matrix Block Structure

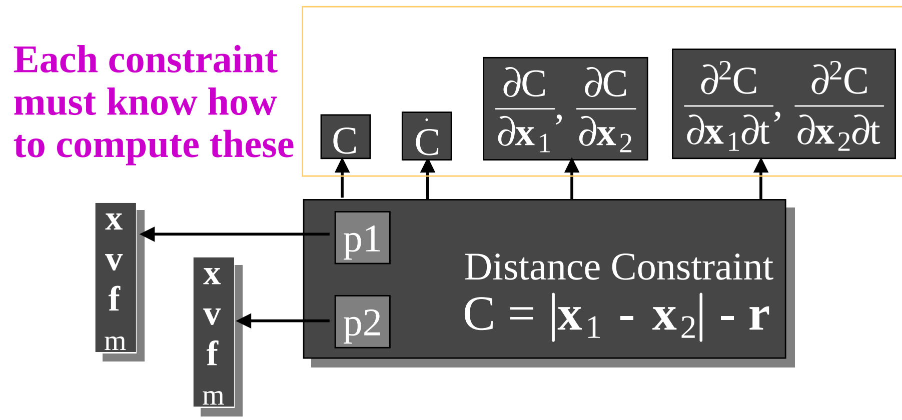


- **Each constraint contributes one or more *blocks* to the matrix**
- **Sparsity: many empty blocks**
- **Modularity: let each constraint compute its own blocks**
- **Constraint and particle indices determine block locations**

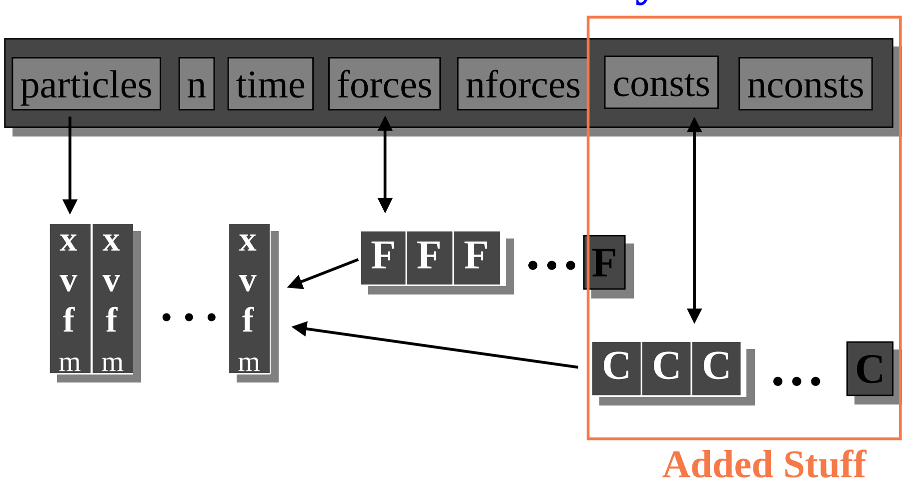
Global and Local



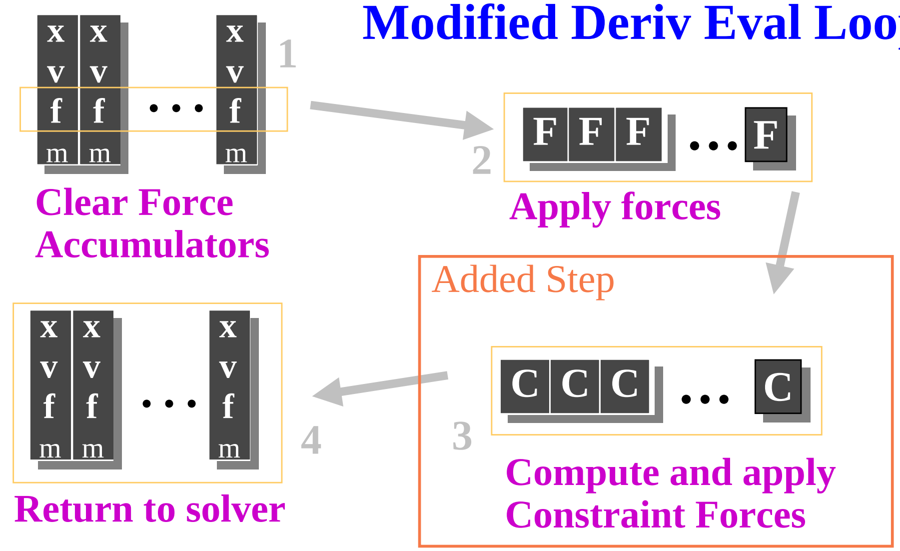
Constraint Structure



Constrained Particle Systems



Modified Deriv Eval Loop



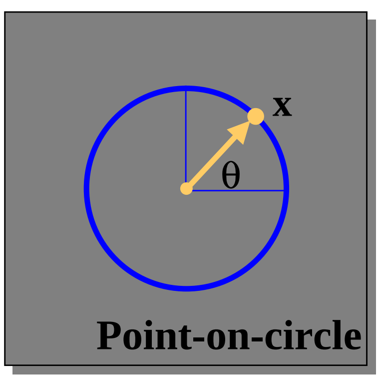
Constraint Force Eval

- After computing ordinary forces:
  - Loop over constraints, assemble global matrices and vectors.
  - Call matrix solver to get λ, multiply by J^T to get constraint force.
  - Add constraint force to particle force accumulators.

Impress your Friends

- The requirement that constraints not add or remove energy is called the *Principle of Virtual Work.*
- The λ's are called *Lagrange Multipliers.*
- The derivative matrix, J, is called the *Jacobian Matrix.*

A whole other way to do it.



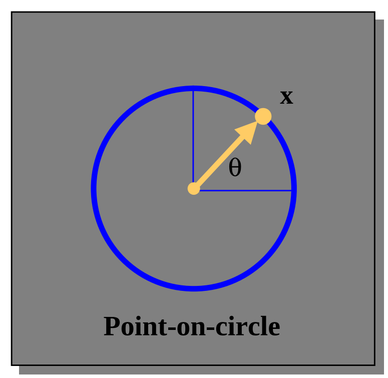
I. Implicit:

~~$C(\mathbf{x}) = |\mathbf{x}| - r = 0$~~

II. Parametric:

$\mathbf{x} = r\,[\cos\theta, \sin\theta]$

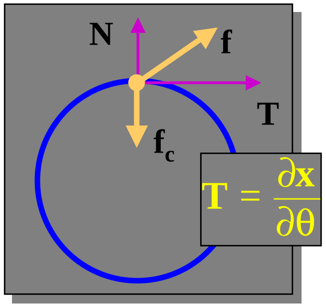
Parametric Constraints



Parametric:
$\mathbf{x} = r\,[\cos\theta, \sin\theta]$

- Constraint is always met exactly.
- One DOF: θ.
- Solve for $\ddot{\theta}$.

Parametric bead-on-wire (f = mv)



x is not an independent variable.

First step—get rid of it:

$$\dot{x} = \frac{f + f_c}{m} \quad \textbf{f = mv } \textit{(constrained)}$$

$$\dot{x} = T\dot{\theta} \quad \textbf{chain rule}$$

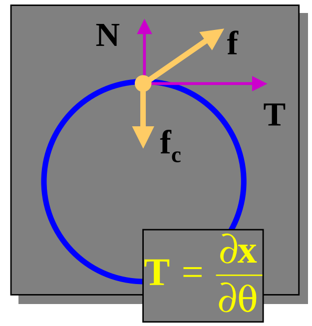
$$T\dot{\theta} = \frac{f + f_c}{m} \quad \textbf{combine}$$

For our next trick…

As before, assume f_c points in the normal direction, so

$\mathbf{T} \cdot \mathbf{f}_c = 0$

We can nuke f_c by dotting T into both sides:



$$T\dot{\theta} = \frac{f + f_c}{m} \quad \textbf{from last slide}$$

$$T \cdot T\dot{\theta} = \frac{T \cdot f + T \cdot f_c}{m} \quad \textbf{blam!}$$

$$\dot{\theta} = \frac{1}{m}\frac{T \cdot f}{T \cdot T} \quad \textbf{rearrange.}$$

General case

Lagrange dynamics:

$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{u}} + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{u}} - \mathbf{J}^T\mathbf{Q} = 0$$

where

$$\mathbf{J} = \frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

Not to be confused with:

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\boldsymbol{\lambda} = -\dot{\mathbf{J}}\dot{\mathbf{q}} - [\mathbf{J}\mathbf{W}]\mathbf{Q}$$

where

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

Parametric Constraints: Summary

- **Generalizations:** $f = ma$**, particle systems**
  - **Like implicit case (see notes)**
- **Big advantages:**
  - **Fewer DOF's**
  - **Constraints are always met**
- **Big *dis*advantages:**
  - **Hard to formulate constraints**
  - **No easy way to *combine* constraints**
- **Offical name: *Lagrangian dynamics***

Hybrid systems

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\boldsymbol{\lambda} = -\dot{\mathbf{J}}\dot{\mathbf{u}} - \left[\mathbf{J}\mathbf{W}\right]\mathbf{Q}$$

where

$$\mathbf{W} = \mathbf{M}^{-1} = \left[\iiint_i m_i \mathbf{q}_i{}^T \mathbf{q}_i\right]^{-1}$$

$$\mathbf{C}(\mathbf{q}(\mathbf{u}))$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

Project 1:

- **A bead on a wire (implicit)**
- **A double pendulum**
- **A *triple* pendulum**
- **Simple interactive tinkertoys**