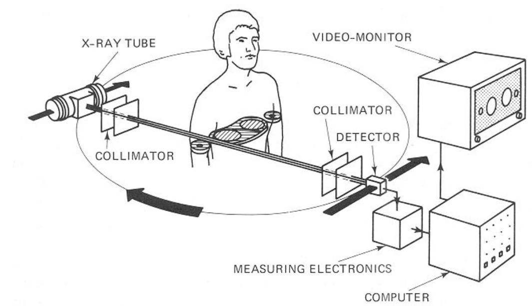
CT Reconstruction

Computerized tomography

Computerized tomography (CT) is a method for using x-ray images to reconstruct a spatially varying density function.

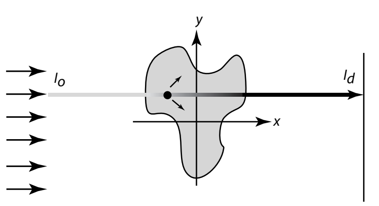


First generation CT scanner

Physics of beam attenuation

CT works by collecting x-ray images one slice at a time.

Consider a parallel beam of x-rays passing through an object being imaged orthographically:



An x-ray photon interacts with the material by:

- absorption
- scatter

Absorbed photons are simply lost.

We will assume that scattered photons are all re-directed away from the sensor.

Physics of beam attenuation

If we consider a single "ray" passing through, we'll find that it's intensity drops off as:

$$\Delta I = -\mu I \Delta x$$

where μ is the **linear attenuation coefficient**.

We can re-write this as differentials and permit μ to vary along the ray:

$$dI = -\mu(x) I dx$$

If the material is made of a single substance of varying density, then μ(x) can be modeled as proportional to that density.

Re-arranging:

$$\frac{dI}{I} = -\mu(x)dx$$

Integrating:

$$\int_{I_o}^{I_d} \frac{dI}{I} = -\int_{-\infty}^{\infty} \mu(x)dx$$

Physics of beam attenuation

Performing the integration of the left side:

$$\int_{I_o}^{I_d} \frac{dI}{I} = \ln[I]\Big|_{I_o}^{I_d} = \ln[I_d] - \ln[I_o] = \ln\left[\frac{I_d}{I_o}\right]$$

Equating to the right side:

$$\ln\left[\frac{I_d}{I_o}\right] = -\int_{-\infty}^{\infty} \mu(x)dx$$

Raising to an exponent:

$$\frac{I_d}{I_o} = \exp\left[-\int_{-\infty}^{\infty} \mu(x)dx\right]$$

Solving for detector intensity:

$$I_d = I_o \exp\left[-\int_{-\infty}^{\infty} \mu(x)dx\right]$$

Considering beams that pass through at various y-positions:

$$I_d(y) = I_o \exp\left[-\int_{-\infty}^{\infty} \mu(x,y)dx\right]$$

Physics of beam attenuation

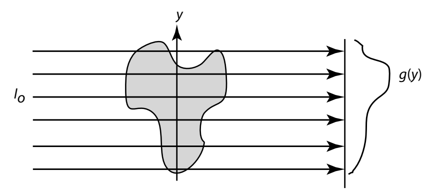
If we back up a little bit, we can remove the negative sign by inverting the argument of the log:

$$\ln\left[\frac{I_o}{I_d}\right] = \int_{-\infty}^{\infty} \mu(x)dx$$

Allowing y to vary:

$$\ln\left[\frac{I_o}{I_d(y)}\right] = \int_{-\infty}^{\infty} \mu(x,y)dx = g(y)$$

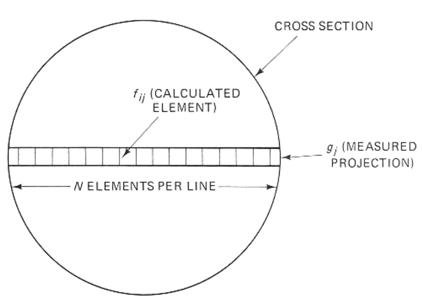
Thus, we can take the detector data, and, using this log, we can interpret the result as an integral projection of the attenuation function.



ART

Using projections from multiple angles, you can try to solve for the interior distribution.

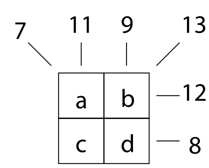
One approach is essentially to create a large linear system and solve iteratively.



Such a technique is called an Algebraic Reconstruction Technique, or ART.

ART

For example:



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 9 \\ 13 \\ 12 \\ 8 \end{bmatrix}$$

In practice, ART has proven computationally expensive and sensitive to noise.

Instead, we can use some fancier math to derive an elegant solution…

The 1D Fourier transform

Recall (from CSE 557?) that the Fourier transform of a 1D function can be written as:

$$\Im_{1D}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\exp[-i2\pi ux]dx = F(u)$$

where u is spatial frequency.

The inverse Fourier transform is simply:

$$\Im_{1D}^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u)\exp[i2\pi ux]du = f(x)$$

Note that an $f(x)$ implies a unique $F(u)$ and vice versa, so if we know one, we can compute the other:

$$f(x) \xrightarrow{\Im} F(u)$$

$$f(x) \xleftarrow{\Im^{-1}} F(u)$$

The 2D Fourier transform

We can generalize this to 2D:

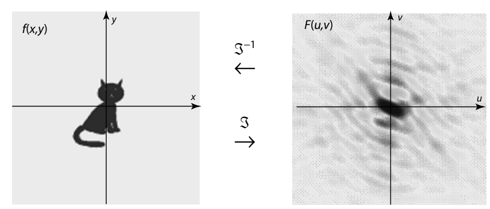
$$\Im_{2D}\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i2\pi(ux+vy)]dxdy = F(u,v)$$

where u is spatial frequency in x, and v is the spatial frequency in y.

Likewise, the 2D inverse Fourier transform is:

$$\Im_{2D}^{-1}\{F(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i2\pi(ux+vy)]dudv = f(x,y)$$

Again, given one function, we can uniquely compute the other.



Linear transforms of Fourier domains

We can also write the Fourier transform relation in terms of vector arguments:

$$f(\mathbf{x}) \xrightarrow{\Im} F(\mathbf{u})$$

It's easy to show that scaling one domain corresponds to inverse scaling the other:

$$f(a\mathbf{x}) \xrightarrow{\Im} \frac{1}{|a|}F\left(\frac{\mathbf{u}}{|a|}\right)$$

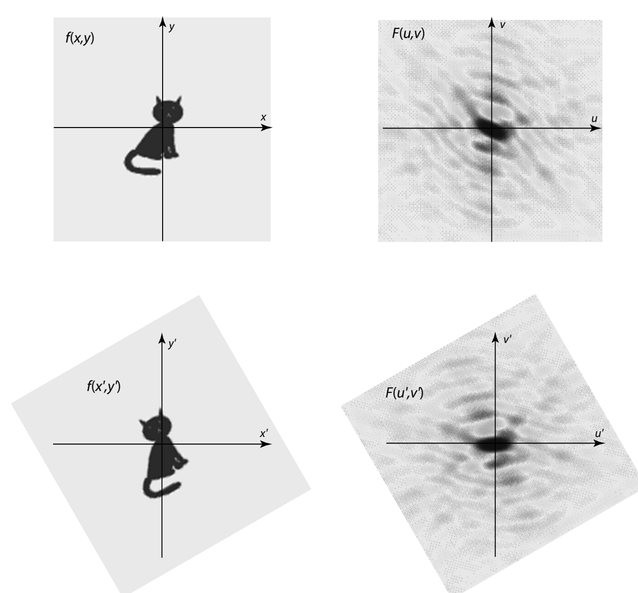
In fact, if we replace "a" with a matrix "A", it is not hard to show that:

$$f(A\mathbf{x}) \xrightarrow{\Im} \left\|A^{-T}\right\| F(A^{-T}\mathbf{u})$$

For rotations, this implies:

$$f(R\mathbf{x}) \xrightarrow{\Im} \left\|R^{-T}\right\| F(R^{-T}\mathbf{u}) = ?$$

Linear transforms of Fourier domains



Fourier transforms and projections

So, what do Fourier transforms have to do with x-ray projections?

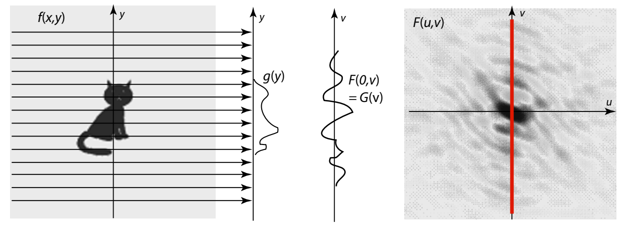
Let's change terminology slightly and say $f(x,y)=\mu(x,y)$. We've already noted that:

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp[-i2\pi(ux+vy)]dxdy$$

What happens if we evaluate this at $F(0,v)$?

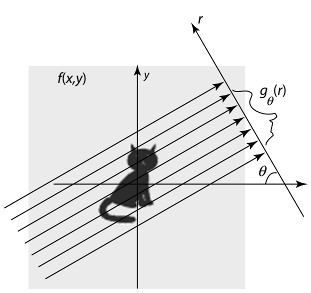
$$\begin{aligned}F(0,v)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp[-i2\pi(u\cdot 0+vy)]dxdy\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp[-i2\pi vy]dxdy\\&=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}f(x,y)dx\right\}\exp[-i2\pi vy]dy\\&=\int_{-\infty}^{\infty}g(y)\exp[-i2\pi vy]dy\\&=\Im_{1D}\{g(y)\}\end{aligned}$$

Fourier transforms and projections

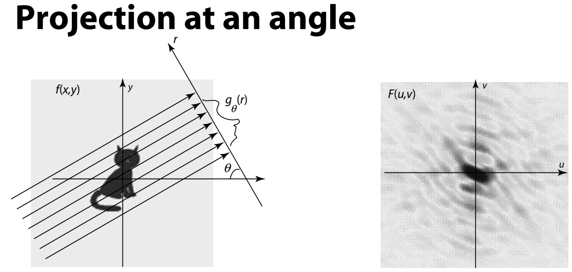


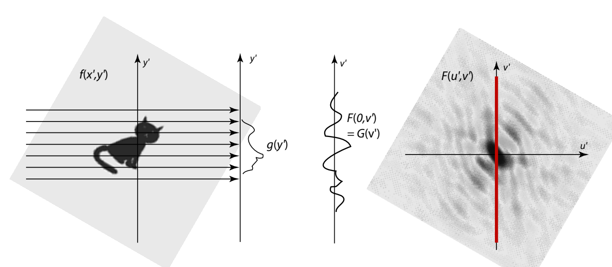
Projection at an angle

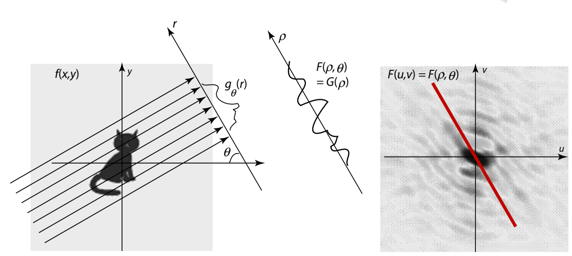
What happens if we project the volume at an angle?



Projection at an angle







Fourier projection slice theorem

In other words, if we express $F(u, v)$ in polar coordinates $F(\rho, \theta)$:

$$F(\rho,\theta) = \Im_{1D}\{g_\theta(r)\} = G_\theta(\rho)$$

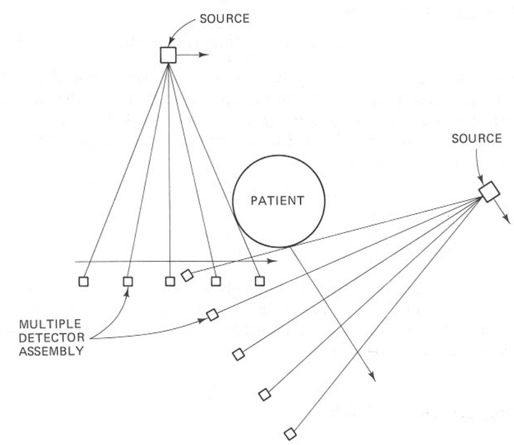
This result is called the "Fourier projection slice theorem" or the "central slice theorem."

Using this theorem, we can reconstruct an object from its projections by:

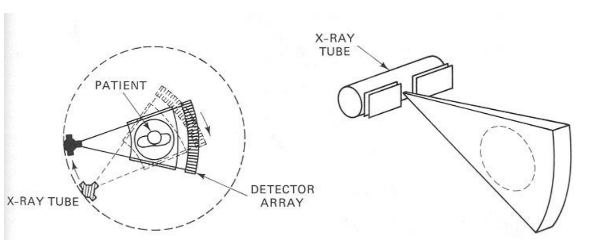
1. Populating the Fourier domain with oriented Fourier lines
2. Taking the inverse Fourier transform

In practice, all of these operations can be performed in the spatial domain.

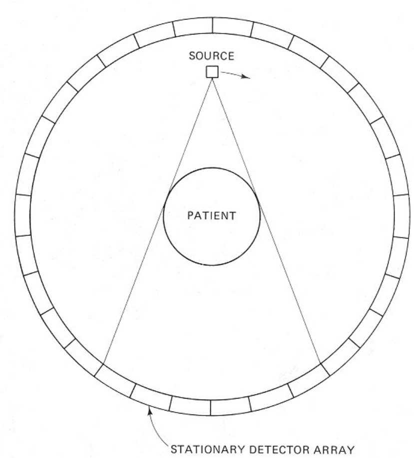
Second generation scanner



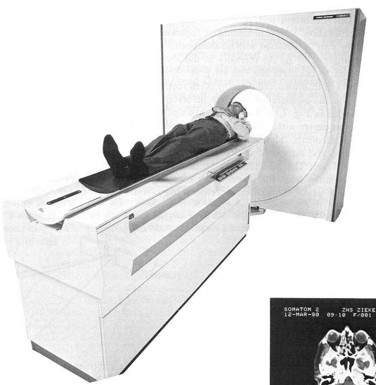
Third generation scanner

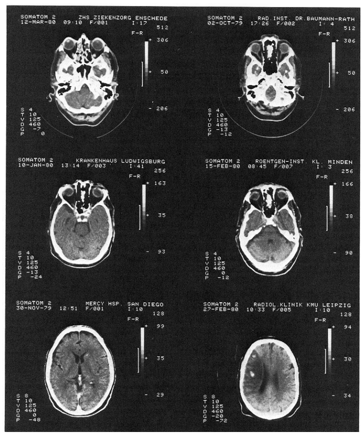


Fourth generation scanner



A medical scanner





Bibliography

Macovski, Medical Imaging Systems, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1983.