15. The radiosity method

Reading

Recommended:

- Cohen and Wallace, *Radiosity and Realistic Image Synthesis*, Chapters 3-5.

The radiosity equation

Assume only diffuse reflection:

- BRDF $\longrightarrow$ reflectance

$$f_r(\mathbf{x}, \omega_\mathbf{i} \rightarrow \omega_\mathbf{o}) = f_{r,d}(\mathbf{x}) = \frac{\rho(x)}{\pi}$$

- Radiance $\longrightarrow$ radiosity

$$L(\mathbf{x}, \omega_\mathbf{o}) = L_o(\mathbf{x}) = \frac{B(\mathbf{x})}{\pi}$$

Starting with the surface rendering equation:

$$L(\mathbf{x}, \omega_\mathbf{o}) = L_e(\mathbf{x}, \omega_\mathbf{o}) + \int_M f_r(\mathbf{x}, \omega_\mathbf{i} \rightarrow \omega_\mathbf{o}) G(\mathbf{x}, \mathbf{x}') L(\mathbf{x}', \omega'_\mathbf{o}(\mathbf{x}, \mathbf{x}')) dA'$$

we drop angular dependences and substitute reflectance and radiosity:

$$\begin{aligned}
L_o(\mathbf{x}) &= L_e(\mathbf{x}) + \int_M f_r(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') L_o(\mathbf{x}') dA' \\
\frac{B(\mathbf{x})}{\pi} &= \frac{B_e(\mathbf{x})}{\pi} + \int_M \frac{\rho(\mathbf{x})}{\pi} G(\mathbf{x}, \mathbf{x}') \frac{B(\mathbf{x}')}{\pi} dA' \\
B(\mathbf{x}) &= B_e(\mathbf{x}) + \int_M \rho(\mathbf{x}) \frac{G(\mathbf{x}, \mathbf{x}')}{\pi} B(\mathbf{x}') dA' \\
B(\mathbf{x}) &= E(\mathbf{x}) + \rho(\mathbf{x}) \int_M F(\mathbf{x}, \mathbf{x}') B(\mathbf{x}') dA'
\end{aligned}$$

The radiosity equation, cont'd

In the last step, we made two substitutions. The first is to replace B_e with E, a common notational choice.

The second is the definition:

$$F(\mathbf{x}, \mathbf{x}') \equiv \frac{G(\mathbf{x}, \mathbf{x}')}{\pi} = \frac{\cos\theta_i \cos\theta'_o}{\pi r^2} V(\mathbf{x}, \mathbf{x}')$$

$F(\mathbf{x}, \mathbf{x}')$ describes the goemetry of light transport between $\mathbf{x}$ and $\mathbf{x}'$.

Overview of the radiosity method

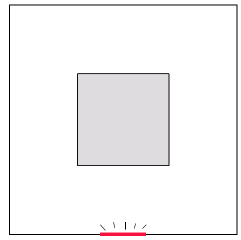
The radiosity method evolved as an idea that was eventually related to the *finite element method.*

Steps:

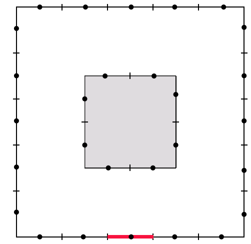
1. Model the environment
2. Discretize the surfaces
   - Subdivide the surfaces
   - Select nodes on elements
3. Choose basis functions
4. Choose a finite error metric ⇒ linear system.
5. Compute coefficients of the matrix — slowest part.
6. Solve the linear system.
7. Reconstruct for display.
8. Display using Gouraud shading — use graphics hardware.

Visualizing radiosity transport

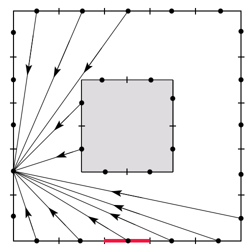
To solve for the radiosity function, we break a scene into patches and solve a light transport equation.



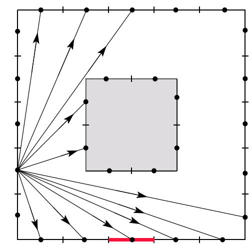
A room containinfg a light and one object



Break up the room into "patches"



Each patch receives light from other patches...

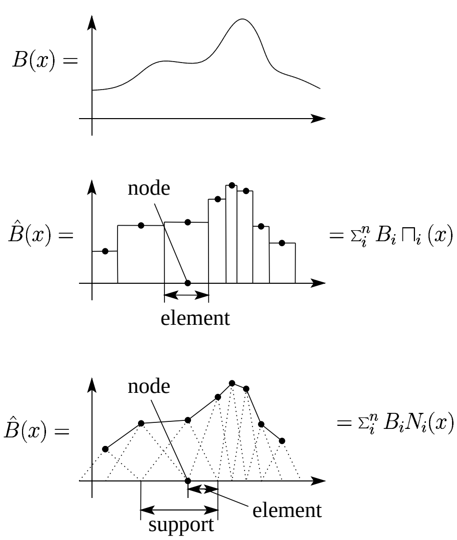


...and reflects light back to these patches.

Basis functions

Given a function $B(\mathbf{x})$, we can subdivide it domain into a finite set of *elements* and *nodes*.

We can then approximate $B(\mathbf{x})$ with a finite set of basis functions, $N_i(\mathbf{x})$, centered at the nodes.



Error metrics

We need to define an error metric for our solution and then choose the B_i that minimize the error.

How about:

$$e(\mathbf{x}) = B(\mathbf{x}) - \widehat{B}(\mathbf{x})$$

Error metrics, cont'd

Instead, we can substitute $\widehat{B}(\mathbf{x})$ into the radiosity equation:

$$\widehat{B}(\mathbf{x}) \;=\; E(\mathbf{x}) + \rho(\mathbf{x}) \int_M F(\mathbf{x},\mathbf{x}')\widehat{B}(\mathbf{x}')dA'$$

In general, there is no solution. Why?

We can define what is called the *residual*:

$$r(\mathbf{x}) = \widehat{B}(\mathbf{x}) - E(\mathbf{x}) - \rho(\mathbf{x}) \int_M F(\mathbf{x},\mathbf{x}')\widehat{B}(\mathbf{x}')dA'$$

Rather than minimize $r(\mathbf{x})$ directly, we integrate against weighting functions and set the integrals to zero:

$$\int r(\mathbf{x})w_i(\mathbf{x})d\mathbf{x} = r_i = 0, i = 1 \cdots n$$

We choose as many weighting functions as we have basis functions. The result is a linear system.

Example weighting functions:

$$\begin{aligned} w_i(\mathbf{x}) &= \delta(\mathbf{x} - \mathbf{x_i}) \\ w_i(\mathbf{x}) &= N_i(\mathbf{x}) \end{aligned}$$

Example

A common choice is:

- Constant basis functions, $N_i(\mathbf{x}) = \sqcap_i(\mathbf{x})$
- Galerkin weighted residuals.

The weighted residuals are then:

$$\begin{aligned} r_i \;=\; & \int_{A_i} \widehat{B}(\mathbf{x}) \sqcap_i (\mathbf{x})dA - \int_{A_i} E(\mathbf{x}) \sqcap_i (\mathbf{x})dA - \\ & \int_{A_i} \rho(\mathbf{x}) \int_M F(\mathbf{x},\mathbf{x}')\widehat{B}(\mathbf{x}')dA' \sqcap_i (\mathbf{x})dA \end{aligned}$$

The first term becomes:

$$\begin{aligned} \int_{A_i} \widehat{B}(\mathbf{x}) \sqcap_i (\mathbf{x})dA &= \int_{A_i} \sum_j B_j \sqcap_j (\mathbf{x}) \sqcap_i (\mathbf{x})dA \\ &= \sum_j B_j \int_{A_i} \sqcap_j(\mathbf{x}) \sqcap_i (\mathbf{x})dA \\ &= \sum_j B_j A_i \delta_{ij} \\ &= B_i A_i \end{aligned}$$

Example, cont'd

If we assume $E(\mathbf{x})$ to be piecewise constant, then the second term becomes:

$$\int_{A_i} E(\mathbf{x}) \sqcap_i (\mathbf{x})dA = E_i A_i$$

If we assume $\rho_i(\mathbf{x})$ to be piecewise constant, then the third term becomes:

$$\begin{aligned} & \int_{A_i} \rho(\mathbf{x}) \int_M F(\mathbf{x},\mathbf{x}')\widehat{B}(\mathbf{x}')dA' \sqcap_i (\mathbf{x})dA = \\ =\; & \int_{A_i} \rho_i \sqcap_i (\mathbf{x}) \int_M F(\mathbf{x},\mathbf{x}') \sum_j B_j \sqcap_j (\mathbf{x})dA' \sqcap_i (\mathbf{x})dA \\ =\; & \rho_i \sum_j B_j \int_{A_i}\int_{A_j} F(\mathbf{x},\mathbf{x}') \sqcap_j (\mathbf{x}) \sqcap_i (\mathbf{x})dA'dA \\ =\; & \rho_i \sum_j B_j \tilde{F}_{ij} \end{aligned}$$

where:

$$\tilde{F}_{ij} = \int_{A_i}\int_{A_j} F(\mathbf{x},\mathbf{x}')dA'dA$$

Example, cont'd

The residual equation now becomes:

$$r_i = B_i A_i - E_i A_i - \rho_i \sum_j B_j \tilde{F}_{ij}$$

Setting $r_i = 0$ gives us:

$$B_i A_i = E_i A_i - \rho_i \sum_j B_j \tilde{F}_{ij}$$

Finally, we divide by A_i and make one last subsitution:

$$B_i = E_i + \rho_i \sum_j B_j F_{ij}$$

Where F_{ij} is the *form factor* between elements A_i and A_j:

$$F_{ij} = \frac{\tilde{F}_{ij}}{A_i} = \frac{1}{A_i} \int_{A_i}\int_{A_j} \frac{\cos\theta_i \cos\theta_o'}{\pi r^2} V(\mathbf{x},\mathbf{x}')dA'dA$$

We can write our finite element version of the radiosity equation in matrix form:

$$\begin{bmatrix} 1-\rho_1 F_{11} & -\rho_1 F_{12} & \cdot\;\cdot\;\cdot\;\cdot & -\rho_1 F_{1n} \\ -\rho_2 F_{21} & 1-\rho_2 F_{22} & & \\ -\rho_3 F_{31} & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \rho_n F_{n1} & & & 1-\rho_n F_{nn} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_n \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ E_n \end{bmatrix}$$

Form factors

The matrix coefficients are comprised of reflectances muliplied by form factors.

The form factor, F_{ij} is the fraction of power per unit area leaving A_i and arriving at A_j.

The form factor *reciprocity relation*:

$$A_i F_{ij} = A_j F_{ji}$$

yields a nice interpretation of the discrete radiosity equation. Multiplying it through by A_i:

$$\begin{aligned}
B_i A_i &= E_i A_i + \rho_i \sum_j B_j A_i F_{ij} \\
B_i A_i &= E_i A_i + \rho_i \sum_j B_j A_j F_{ji} \\
\Phi_o &= \Phi_e + \rho \sum_j (\Phi_o)_j F_{ji} \\
\Phi_o &= \Phi_e + \rho \Phi_i
\end{aligned}$$

Computing form factors

There are two basic approaches to computing form factors:

- Analytic
  - There are exact solutions for some geometries.
  - Still have to solve for occlusions...
- Quadrature
  - Sample the integral, including occlusions
  - Regular vs. stochastic sampling

Solving the linear system

The linear system is of a form:

$$MB = E$$

This can also be written as:

$$(I - T)B = E$$

Where T is the discrete radiosity transport operator.

Two important properties of M and T:

1. M and T are not sparse.
2. Eigenvalues of T are less than 1.

Direct solution methods:

- Compute $B = M^{-1}E$.
- Inefficient for large systems: $O(n^3)$

Iterative solution methods

- Update the solution in steps that gradually improve the solution.
- Properties of M ensure convergence.
- Ideal for graphics: a few iterations can yield an adequate image.

Jacobi iteration

Example: Jacobi iteration.

Solution is the Neumann series:

$$B = (I - T)^{-1} = E + TE + T^2E + \cdots$$

This can be solved in steps:

$$\begin{aligned}
B^{(0)} &\leftarrow E \\
B^{(1)} &\leftarrow B^{(0)} + TB^{(0)} \\
B^{(2)} &\leftarrow B^{(1)} + TB^{(1)} \\
&\vdots
\end{aligned}$$

Truncate after m iterations: $O(mn^2)$.

Jacobi iteration, cont'd

Cost of Jacobi iteration:

- $O(n^2)$ interactions
- $O(n^2)$ form factors to compute
- $O(n^2 \log n)$ to compute form factors
- $O(mn^2)$ to solve

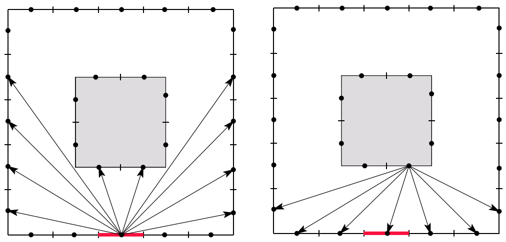
Bottom line: must compute $O(n^2)$ form factors.

Other methods are much faster still...

Progressive radiosity

Whenever we multiply a row of T by B, we can think of this as gathering energy into a patch.

A faster converging method is based on the notion of "shooting" energy from patches.



Progressive radiosity, cont'd

The idea is:

1. Sort the patches by the amount of radiosity they currently have.
2. Shoot the energy from the brightest patch to all the other patches.
3. Mark this patch as having zero "unshot" radiosity.
4. Choose the next patch with the largest unshot radiosity and iterate.

This approach is called "progressive radiosity".

The convergence is substantially faster, closer to $O(n)$ in practice.

Hierarchical radiosity

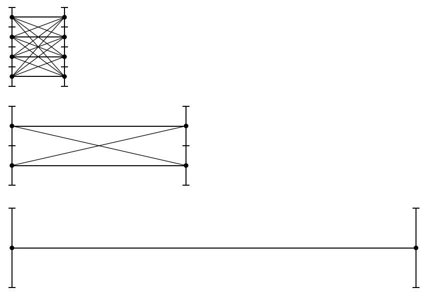
In many cases, there is an inherent imbalance in the element to element interactions:

Inspiration for a solution: fast N-body algorithms.

Hierarchical radiosity, cont'd

Refinement algorithm:

1. Estimate visibility, V_{ij}
2. If V_{ij} not 1 or 0, subdivide larger patch and go to 1.
3. Estimate F_{ij}, F_{ji}
4. If $F_{ij}, F_{ji} > F_\epsilon$, subdivide larger patch and go to 1.
5. "Link" patches



Hierarchical radiosity example

Solving the hierarchical system

Until converged

For each patch

Gather energy for each patch and its children

Distribute energy up and down the patch's hierarchy

Can incorporate refinement into the soltuion stage:

- Compute BF energy transports and refine links only if significant.

The complexity can be shown to be linear in n, the number of leaf node patches.

However, the original input patches are the parent patches, so if there are k input patches, there are at least $O(k^2)$ interactions. Total complexity is then $O(n + k^2)$.

Reconstruction and display

For rendering with graphics hardware, must compute radiosity at the vertices:

The result is a view-independent solution. Can display with fast Gouraud shading.

Meshing

Choosing the mesh subdivision affects results significantly.

Problems:

1. Light leaks

2. Shadows

Solution is called "discontinuity meshing."

Radiosity and view-*dependence*

The Gouraud-shaded result exhibits artifacts and does not model specularity.

1. Final gather

Compute view-independent solution. Then:

- Cast a ray through each pixel
- Intersect with surface patch.
- Cast rays to each linked patch.
- Add up the radiosity and return.

Radiosity and view-*dependence*, cont'd

2. Importance

Do hierarchical refinement based on contribution to current viewpoint.

E.g., subdivide visible surfaces and their linked counterparts more finely.

3. Multi-pass methods

Basic idea:

- Use radiosity to capture diffuse interreflections.
- Use ray tracing to capture specular components.