13a. Radiometry...the rendering equation

Reading

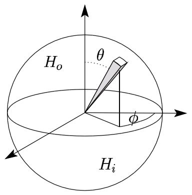
Recommended:

- Cohen and Wallace, *Radiosity and Realistic Image Synthesis*, Chapter 2.

Irradiance

The irradiance is the power per unit area incident on a surface. Equivalent to flux density.

$$E = \int_{H_i} L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_i$$

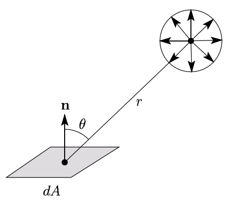


Radiant intensity

The radiant intensity is the power per unit solid angle from a point.

$$I(\omega) = \frac{d\Phi}{d\omega}$$

Irradiance due to an isotropic point source:



Radiosity

The radiosity is the energy per unit area leaving a surface:

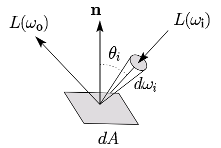
$$B = \int_{H_o} L(\omega_{\mathbf{o}}) \cos\theta_o d\omega_o$$

Officially referred to as the radiant exitance.

Consider a uniform diffuse source:

The BRDF

The Bidirectional Reflectance Distribution Function measures the ratio of differential outgoing radiance to differential incoming radiance.



$$\begin{aligned} f_r(\omega_{\mathbf{i}} \to \omega_{\mathbf{o}}) &= \frac{dL(\omega_{\mathbf{o}})}{dE(\omega_{\mathbf{i}})} \\ &= \frac{dL(\omega_{\mathbf{o}})}{L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_i} \end{aligned}$$

Properties:

- Helmholtz reciprocity

$$f_r(\omega_{\mathbf{i}} \to \omega_{\mathbf{o}}) = f_r(\omega_{\mathbf{o}} \to \omega_{\mathbf{i}})$$

- Isotropic vs. anisotropic

$$f_r((\theta_i, \phi_i) \to (\theta_o, \phi_o)) \stackrel{?}{=} f_r((\theta_i, \phi_i + \Delta\phi) \to (\theta_o, \phi_o + \Delta\phi))$$

The reflection equation

Outgoing radiance is computed by integrating over incoming radiance:

$$L(\omega_{\mathbf{o}}) = \int_{H_i} f_r(\omega_{\mathbf{i}} \to \omega_{\mathbf{o}}) L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_{\mathbf{i}}$$

Linearity:

- Doubling the incident light doubles the reflected light.
- Reflection from several sources behaves like the sum of reflections from each source.

Examples of reflection

Ideal specular:

$$f_r(\omega_{\mathbf{i}} \to \omega_{\mathbf{o}}) \approx \delta(\omega_{\mathbf{o}} - \omega_{\mathbf{r}}(\omega_{\mathbf{i}}))$$

$$\begin{aligned} L(\omega_{\mathbf{o}}) &\approx \int_{H_i} \delta(\omega_{\mathbf{o}} - \omega_{\mathbf{r}}(\omega_{\mathbf{i}})) L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_{\mathbf{i}} \\ &\approx L(\omega_{\mathbf{r}}(\omega_{\mathbf{o}})) \end{aligned}$$

Ideal diffuse:

$$f_r(\omega_{\mathbf{i}} \to \omega_{\mathbf{o}}) = f_{r,d}$$

$$\begin{aligned} L(\omega_{\mathbf{o}}) &= \int_{H_i} f_{r,d} L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_{\mathbf{i}} \\ &= f_{r,d} \int_{H_i} L(\omega_{\mathbf{i}}) \cos\theta_i d\omega_{\mathbf{i}} \\ &= \end{aligned}$$

Bihemsipherical reflectance

For diffuse surfaces, it is useful to compute a ratio of total outgoing flux density to total incoming flux density.

This ratio is the *bihemsipherical reflectance*, ρ.

For diffuse surfaces:

$$\begin{aligned}\rho &= \frac{B}{E} \\ &= \frac{\pi L(\omega_o)}{E} \\ &= \pi f_{r,d}\end{aligned}$$

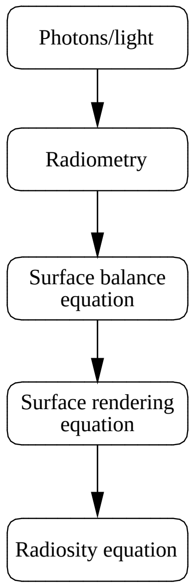
By conservation of energy, $0 \leq \rho \leq 1$.

The grand scheme

Physics
  Transport theory

Mathematics
  Integral equations

Computer Science
  Algorithms
  Numerics



Surface balance equation

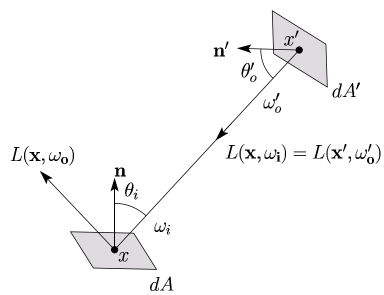
$$\begin{array}{ccccccc}[\text{outgoing}] & = & [\text{emitted}] & + & [\text{reflected}] & + & [\text{transmitted}] \\ L_o(\mathbf{x},\omega) & = & L_e(\mathbf{x},\omega) & + & L_r(\mathbf{x},\omega) & + & L_t(\mathbf{x},\omega)\end{array}$$

1. Ignore transmission (easy to include it again later)

2. Reflection equation related incident to reflected radiance:

$$L(\mathbf{x},\omega_\mathbf{o}) = L_e(\mathbf{x},\omega_\mathbf{o}) + \int_{H_i} f_r(\mathbf{x},\omega_\mathbf{i} \rightarrow \omega_\mathbf{o}) L(\mathbf{x},\omega_\mathbf{i}) \cos\theta_i d\omega_\mathbf{i}$$

Two point geometry



$$\omega_\mathbf{i} = \omega'_\mathbf{o} = \frac{\mathbf{x} - \mathbf{x'}}{\| \mathbf{x} - \mathbf{x'} \|}$$

$$d\omega_i = \frac{\cos\theta'_o dA'}{\| \mathbf{x} - \mathbf{x'} \|^2}$$

The rendering equation

Our objective is to solve for the radiance function, $L(\mathbf{x}, \omega)$ over all surfaces in all directions.

To do this, we must generalize the surface reflection equation to include the presence of other surfaces.

The result is the "rendering equation," which we will write out in three different ways...

1. Directional integral

$$L(\mathbf{x}, \omega_\mathbf{o}) = L_e(\mathbf{x}, \omega_\mathbf{o}) + \int\limits_{H_i} f_r(\mathbf{x}, \omega_\mathbf{i} \to \omega_o) L(\mathbf{x}'(\mathbf{x}, \omega_\mathbf{i}), \omega_\mathbf{i}) \cos\theta_i d\omega_\mathbf{i}$$

The rendering equation, cont'd

2. Surface integral

Noting that:

$$\cos\theta_i d\omega_i = \frac{\cos\theta_i \cos\theta'_o}{\| \mathbf{x} - \mathbf{x}' \|^2} dA'$$

we obtain:

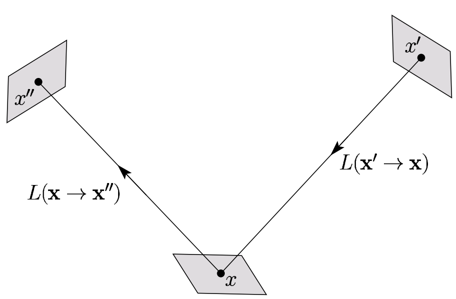
$$L(\mathbf{x}, \omega_\mathbf{o}) = L_e(\mathbf{x}, \omega_\mathbf{o}) + \int\limits_{M} f_r(\mathbf{x}, \omega_\mathbf{i} \to \omega_o) G(\mathbf{x}, \mathbf{x}') L(\mathbf{x}', \omega'_\mathbf{o}(\mathbf{x}, \mathbf{x}')) dA'$$

where the integral is over all surfaces (the set of 2D manifolds, M), and:

$$G(\mathbf{x}, \mathbf{x}') = \frac{\cos\theta_i \cos\theta'_o}{\| \mathbf{x} - \mathbf{x}' \|^2} V(\mathbf{x}, \mathbf{x}')$$

The rendering equation, cont'd

3. Three point transport (a variant on the surface integral)



$$L(\mathbf{x} \to \mathbf{x}'') = L_e(\mathbf{x} \to \mathbf{x}'') + \int\limits_{M} f_r(\mathbf{x}' \to \mathbf{x} \to \mathbf{x}'') G(x, x') L(\mathbf{x}' \to \mathbf{x}) dA'$$

Interpreting the rendering equation

Here is a simplified version of the surface rendering equation:

$$L(\mathbf{x}, \omega) = L_e(\mathbf{x}, \omega) + \int \tau(\mathbf{x}, \omega' \to \omega) L_e(\mathbf{x}', \omega') dA'$$

This follows the form of equation:

$$f(u) + g(u) + \int \kappa(u, v) f(v) dv$$

known as a *Fredholm integral equation of the second kind.*

Interpreting the rendering equation, cont'd

Let's define an integral operator:

$$(\mathcal{T}L)(\mathbf{x},\omega) = \int \tau(\mathbf{x},\omega' \to \omega)L_e(\mathbf{x}',\omega')dA'$$

We are then solving:

$$L = L_e + \mathcal{T}L \Rightarrow (I - \mathcal{T})L = L_e \Rightarrow L = (I - \mathcal{T})^{-1}L_e$$

A more intuitive way of writing this is:

$$\begin{aligned} L &= L_e + \mathcal{T}L \\ &= L_e + \mathcal{T}(L_e + \mathcal{T}L) \\ &= L_e + \mathcal{T}(L_e + \mathcal{T}(L_e + \cdots \\ &= L_e + \mathcal{T}L_e + \mathcal{T}^2L_e + \mathcal{T}^3L_e + \cdots \end{aligned}$$

Solving the rendering equation

We can solve the rendering equation by:

- Sampling (ray tracing).
- Discretizing the space of surfaces and directions (finite elements, linear algebra).
  - If we assume diffuse surfaces, then the directional variation vanishes and we arrive at the "radiosity method."