5. Transformations

Reading

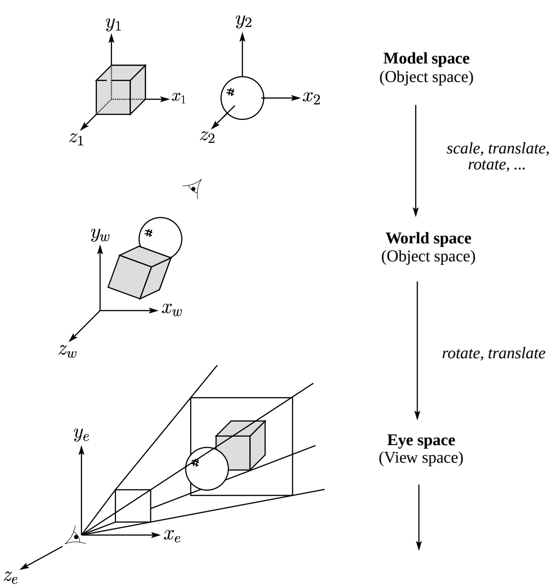
Required:

- Foley *et al.*, Chapter 5.6 and Chapter 6

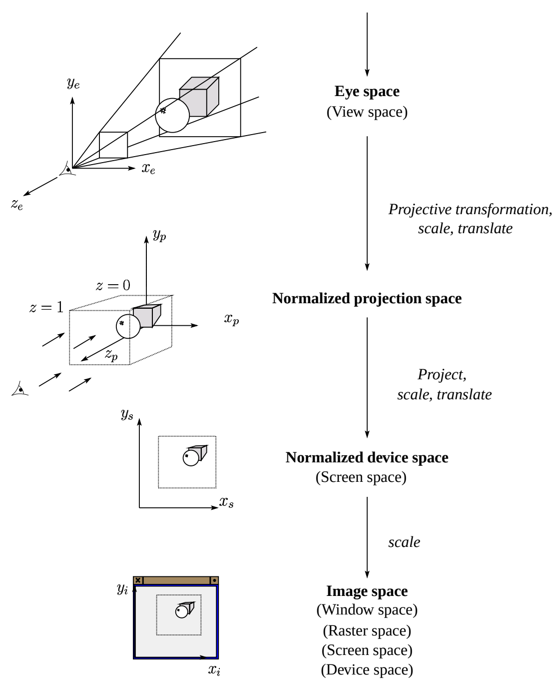
Supplemental:

- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, Second edition, McGraw-Hill, New York, 1990, Chapter 3.

3D Geometry Pipeline



3D Geometry Pipeline (cont'd)



Types of transformations

Continuous (preserves neighborhoods)

One-to-one (invertible)

We can classify them by invariants or symmetries:

- Isometry (distance preserved)
  Reflections, rigid body motions (rotations/translations)
- Similarity (preserves angles)
  Uniform scale
- Affine (preserves parallel lines)
  Non-uniform scales, shears
- Collineation (lines remain lines)
  Perspective
- Non-linear (lines become curves)
  Twists, bends, warps, morphs

Types of transformations

Another way to group transformations is in terms of their matrix structure. Among these, we will consider linear, affine, and projective transformations.

These transformations subsume the most common transformations in graphics:

- Scaling
- Rotation
- Shearing
- Translation
- Projection

Linear transformations

Linear transformations map vectors to vectors:

$$F : \Re^n \to \Re^m$$

We have already seen the properties needed for linearity. Succinctly:

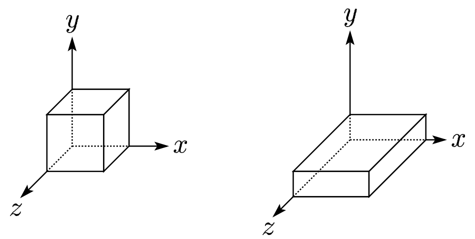
$$F\left(\sum_i \alpha_i \mathbf{v_i}\right) = \sum_i \alpha_i F\left(\mathbf{v_i}\right)$$

Linear transformations of vectors can be expressed in terms of matrix multiplication:

$$F(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

Q: Why isn't translation linear under this definition?

Scaling



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} \mathbf{S}(a,b,c)\mathbf{S}(a',b',c') &= \mathbf{S}(aa',bb',cc') \\ \mathbf{S}(a,b,c)\mathbf{S}(a',b',c') &= \mathbf{S}(a',b',c')\mathbf{S}(a,b,c) \\ \mathbf{S}^{-1}(a,b,c) &= \mathbf{S}(1/a,1/b,1/c) \end{aligned}$$

Rotation

$$\mathbf{R_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R_y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

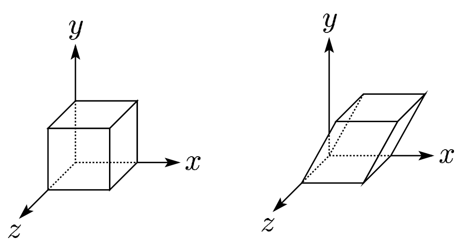
$$\mathbf{R_z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotation, (cont'd)

$$\begin{aligned} \mathbf{R_a}(0) &= \mathbf{I} \\ \mathbf{R_a}(\theta)\mathbf{R_a}(\phi) &= \mathbf{R_a}(\theta+\phi) \\ \mathbf{R_a}(\theta)\mathbf{R_a}(\phi) &= \mathbf{R_a}(\phi)\mathbf{R_a}(\theta) \\ \mathbf{R_a}(\theta)\mathbf{R_b}(\phi) &\neq \mathbf{R_b}(\phi)\mathbf{R_a}(\theta) \\ \mathbf{R_a}^{-1}(\theta) &= \mathbf{R_a}(-\theta) = \mathbf{R_a}^T(\theta) \end{aligned}$$

Q: How many parameters describe a rotation?

Shearing



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Affine transformations

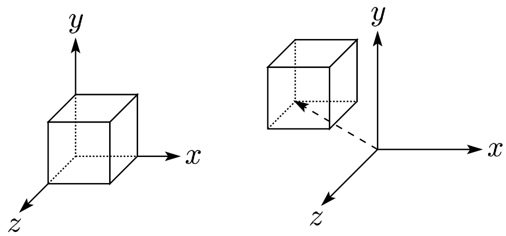
Afine transformations are essentially linear transformations plus translations:

$$F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

We can re-write this as a "linear transformation on an affine space" using an additional coordinate:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{T}(\mathbf{0}) &= I \\ \mathbf{T}(\mathbf{u})\mathbf{T}(\mathbf{v}) &= \mathbf{T}(\mathbf{u}+\mathbf{v}) \\ \mathbf{T}(\mathbf{u})\mathbf{T}(\mathbf{v}) &= \mathbf{T}(\mathbf{v})\mathbf{T}(\mathbf{u}) \\ \mathbf{T}^{-1}(\mathbf{v}) &= \mathbf{T}(-\mathbf{v}) \end{aligned}$$

Affine geometry

Basic geomtric objects:

- Points: location in 3D space
- Vectors: quantity with a direction and magnitude, but no fixed position.
- Scalar: a real number, a unit of measurement

Points and vectors exist independently of any specific reference system.

Affine spaces

An *affine space* consists of points and vectors related by a set of axioms.

Axioms:

- The difference of two points is a vector:

$$\begin{aligned} \mathbf{v} &= P - Q \\ P &= Q + \mathbf{v} \end{aligned}$$

- Head-to-tail rule for vector addition:

$$\begin{array}{ccccc} (P-Q) & + & (Q-R) & = & P-R \\ \mathbf{u} & + & \mathbf{v} & = & \mathbf{w} \end{array}$$

Affine operations

Here is the set of "legal" affine operations:

$$\begin{array}{ccccc} \text{vector} & + & \text{vector} & \rightarrow & \text{vector} \\ \text{scalar} & \cdot & \text{vector} & \rightarrow & \text{vector} \\ \text{point} & - & \text{point} & \rightarrow & \text{vector} \\ \text{point} & + & \text{vector} & \rightarrow & \text{point} \\ \text{point} & + & \text{point} & \rightarrow & \textbf{nonsense} \end{array}$$

...and an example of an "illegal" operation:

$$\text{point} + \text{point} \rightarrow \textbf{nonsense}$$

One useful combination of affine operations is:

$$P(\alpha) = P_o + \alpha\mathbf{v}$$

What does this describe?

Affine combination

An affine combination of two points is:

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2$$

where $\alpha_1 + \alpha_2 = 1$. Why is this permissible?

We can use the scalar values to define:

$$\text{ratio}(Q_1, Q, Q_2) = \frac{\overline{QQ_2}}{\overline{Q_1 Q}} = \frac{\alpha_1}{\alpha_2}$$

We can generalize the notion of affine combination to multiple points:

$$Q = \alpha_1 Q_1 + \cdots + \alpha_k Q_k$$

where $\Sigma\, \alpha_i = 1$.

Frame for an affine space

A frame can be defined in terms of a set of vectors and a point:

$$(\mathbf{v_1}, \cdots, \mathbf{v_n}, \mathcal{O})$$

where $\mathbf{v_1}, \cdots, \mathbf{v_n}$ form a basis and $\mathcal{O}$ is a point in space.

Any point P can then be written as:

$$P = p_1 \mathbf{v_1} + \cdots + p_n \mathbf{v_n} + \mathcal{O}$$

and any vector as:

$$\mathbf{u} = u_1 \mathbf{v_1} + \cdots + u_n \mathbf{v_n}$$

Barycentric coordinates

A set of points can be used to creat a frame.

$$\begin{aligned} P &= \mathcal{O} + p_1 \mathbf{v_1} + \cdots + p_n \mathbf{v_n} \\ Q_o &= \mathcal{O} \\ Q_i &= \mathcal{O} + \mathbf{v_i} \\ \Rightarrow P &= p_o Q_o + p_1 Q_1 + \cdots p_n Q_n \end{aligned}$$

where $p_o = 1 - (p_1 + \cdots p_n)$.

We say that $(p_o, \cdots, p_n)$ are the barycentric coordinates of P relative to $Q_o, \cdots, Q_n$.

Vectors in barycentric coordinates

Similarly for vectors...

$$\begin{aligned} \mathbf{u} &= u_1 \mathbf{v_1} + \cdots + u_n \mathbf{v_n} \\ Q_o &= \mathcal{O} \\ Q_i &= \mathcal{O} + \mathbf{v_i} \\ \Rightarrow \mathbf{u} &= u_o Q_o + u_1 Q_1 + \cdots u_n Q_n \end{aligned}$$

where $u_o = -(u_1 + \cdots u_n)$, i.e., $\Sigma\, u_i = 0$.

Euclidean and Cartesion spaces

A Euclidean space is an affine space with an inner product: $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

A Cartesian space is a Euclidean space with a standard orthonormal frame. In 3D: $(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathcal{O})$.

Some useful properties and operations in Cartesian spaces:

$$\begin{aligned} \text{Length:} \quad & |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \\ \text{Distance between points:} \quad & |P - Q| \\ \text{Angle between vectors:} \quad & \cos^{-1}\left(\tfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) \\ \text{Perpendicular (orthogonal):} \quad & \mathbf{u} \cdot \mathbf{v} = 0 \\ \text{Parallel:} \quad & \tfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \pm 1 \\ \text{Cross product (in 3D):} \quad & \mathbf{u} \times \mathbf{v} = \mathbf{w} \end{aligned}$$

Properties of affine transformations

$F : A \to B$ is an affine transformation if it preserves affine combinations:

$$F(\sum \alpha_i Q_i) = \sum \alpha_i F(Q_i)$$

where $\sum \alpha_i = 1$. A similar results obtains for affine combinations of vectors.

Affine coordinates are preserved:

$$F(\mathcal{O} + \sum p_i \mathbf{v_i}) = F(\mathcal{O}) + \sum p_i F(\mathbf{v_i})$$

Lines map to lines:

$$F(P_o + \alpha \mathbf{v}) = F(P_o) + \alpha F(\mathbf{v})$$

Parallelism is preserved:

$$F(Q_o + \beta \mathbf{v}) = F(Q_o) + \beta F(\mathbf{v})$$

Ratios are preserved:

$$\text{Ratio}(Q_1, Q, Q_2) = \text{Ratio}(F(Q_1), F(Q), F(Q_2))$$