3a. More sampling theory

Digital filtering

Let's consider the case where we want to filter a sampled signal and then resample it.

We can write this as:

where $r(x)$ is the reconstruction filter and $h(x)$ is the filter we wish to apply.

Digital filtering, cont'd

We can show that convolution is associative:

$$(a * b) * c = a * (b * c)$$

so that:

$$\begin{aligned} g(x) &= \{[f(x) \cdot \mathrm{III}(x)] * (r(x) * h(x))\} \, \mathrm{III}(x) \\ &= \left\{[f(x) \cdot \mathrm{III}(x)] * \tilde{h}(x)\right\} \mathrm{III}(x) \end{aligned}$$

where $\tilde{h}(x) = r(x) * h(x)$.

Digital filtering, cont'd

Now let's start building the terms:

$$\begin{aligned} f(x)\mathrm{III}(x) &= \sum_i f(i)\delta(x-i) \\ &\equiv \sum_i f[i]\delta(x-i) \\ [f(x)\mathrm{III}(x)] * \tilde{h}(x) &= \sum_i f[i]\tilde{h}(x-i) \\ \left\{[f(x)\mathrm{III}(x)] * \tilde{h}(x)\right\} \mathrm{III}(x) &= \sum_k \sum_i f[i]\tilde{h}(x-i)\delta(x-k) \\ &= \sum_k \sum_i f[i]\tilde{h}[k-i]\delta(x-k) \\ &= \sum_k \{f[k] * h[k]\} \, \delta(x-k) \end{aligned}$$

We now have the notion of "digital" or "discrete" convolution:

$$g[k] = f[k] * h[k] \equiv \sum_i f[i]\tilde{h}[k-i]$$

2D Fourier transform

The Fourier transform generalizes into 2D as:

$$
\begin{aligned}
F(s_x, s_y) &= \mathcal{F}_y\left\{\mathcal{F}_x\left\{f(x,y)\right\}\right\} \\
&= \mathcal{F}_y\left\{\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi s_x x}\,dx\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi s_x x}e^{-j2\pi s_y y}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(s_x x + s_y y)}\,dx\,dy
\end{aligned}
$$

and the inverse 2D Fourier transform:

$$
f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s_x, s_y)e^{j2\pi(s_x x + s_y y)}\,ds_x\,ds_y
$$

Separable functions

For separable functions, $f(x,y) = f_x(x)f_y(y)$, so that:

$$
\begin{aligned}
F(s_x, s_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x)f_y(y)e^{-j2\pi(s_x x + s_y y)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x)f_y(y)e^{-j2\pi s_x x}e^{-j2\pi s_y y}\,dx\,dy \\
&= \int_{-\infty}^{\infty} f_x(x)e^{-j2\pi s_x x}\,dx \int_{-\infty}^{\infty} f_y(y)e^{-j2\pi s_y y}\,dy \\
&= \mathcal{F}_x\left\{f_x(x)\right\}\mathcal{F}_y\left\{f_y(y)\right\} \\
&= F_{s_x}(s_x)F_{s_y}(s_y)
\end{aligned}
$$

Separable function example

For example:

$$
f(x,y) = \sqcap(x)\sqcap(y)
$$

yields:

$$
F(s_x, s_y) =
$$

2D sampling

We can also define a 2D impulse train as:

$$
\mathrm{III}(x,y) = \sum_i \sum_k \delta(x-i, y-k)
$$

As in 1D, the 2D impulse train is its own Fourier transform:

$$
\mathrm{III}(x,y) \xrightarrow{\mathcal{F}} \mathrm{III}(s_x, s_y)
$$

2D sampling, cont'd

If the Fourier transform of $f(x, y)$ is:

then sampling and reconstruction yield:

Convolution in 2D

In two dimensions, convolution becomes:

$$\begin{aligned} h(x, y) &= f(x, y) * g(x, y) \\ &= \int \int_{-\infty}^{\infty} f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta \end{aligned}$$

Similarly, discrete convolution in 2D is:

$$\begin{aligned} h[i, k] &= f[i, k] * g[i, k] \\ &= \sum_{m} \sum_{l} f[m, l] g[i - m, k - l] \end{aligned}$$