3. Sampling theory

Reading

Strongly Recommended:

- Foley, et al, section 14.10.

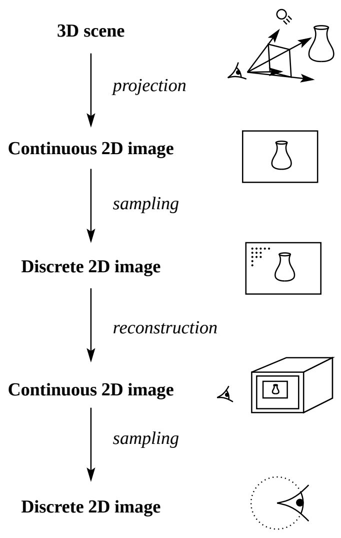
Recommended:

- Don Mitchell and Arun Netravali, Reconstruction Filters in Computer Graphics. *Computer Graphics* (SIGGRAPH '88) 22(4), 221-228.

Samples in graphics

In computer graphics, we encounter sampled representations constantly.

Example: the rendering pipeline.



Samples in graphics, cont'd

Let's list a number of examples of samples in graphics:

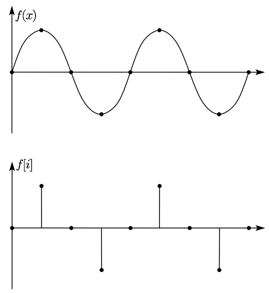
Aliasing in graphics

One of the most objectionable artifacts that arises in graphics is *aliasing*.

Consider a continuous function $f(x)$. Now sample it at intervals Δ to give $f[i] = f(i\Delta)$.

Q: How well does $f[i]$ approximate $f(x)$?

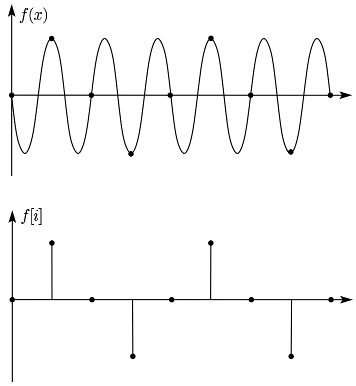
Consider sampling a sinusoid:



In this case, the sinusoid is reasonably well approximated by the samples.

Aliasing, cont'd

Now consider sampling a higher frequency sinusoid

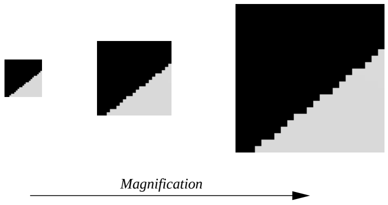


We get the exact same samples, so we seem to be approximating the first lower frequency sinusoid again.

We say that, after sampling, the higher frequency sinusoid has taken on a new "alias", i.e., changed its identity to be a lower frequency sinusoid.

Aliasing, cont'd

Other examples include aliasing due to polygon rasterization:



Temporal aliasing:

Linear shift-invariant systems

To study the theory of sampling and reconstruction, we need some definitions, starting with linear systems.

$$g = L[f]$$

L is *linear* if:

$$\begin{aligned} L[f_1 + f_2] &= L[f_1] + L[f_2] \\ L[af] &= aL[f] \end{aligned}$$

L is *shift-invariant* if:

$$L[f(x-a)] = g(x-a)$$

Convolution

The behavior of a Linear Shift-Invariant (LSI) system can be written in terms of *convolution*:

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(\alpha) h(x-\alpha)\, d\alpha \\ g &= f * h \\ g &= h * f \end{aligned}$$

where $h(x)$ is the *impulse response*. (The choice of terminology will be clearer shortly.)

To visualize this, let's consider a symmetric (a.k.a., even) function, $h(x) = h(-x)$. Then:

$$g(x) = \int_{-\infty}^{\infty} f(\alpha) h(\alpha - x)\, d\alpha$$

Convolution with the rect function

For example, the "rect" function, $\sqcap(x)$:

$$h(x) = \sqcap(x) = \begin{cases} 0 & : \; x > \frac{1}{2} \\ \frac{1}{2} & : \; x = \frac{1}{2} \\ 1 & : \; x < \frac{1}{2} \end{cases}$$

Convlution means sliding $\sqcap(x)$ over $f(x)$ and computing the integral at each position:

In this case, $h(x)$ is a *filter* that averages over a neighborhood and smooths $f(x)$.

The impulse function

The most important function in sampling theory is the *impulse function* or *Dirac delta function* or just *delta function*.

It is defined in the limit:

such that:

$$\int \delta(x)\, dx = 1$$

It is drawn as:

Sifting and shifting

Since the delta function is zero everywhere except where its argument is zero, we can derive the *sifting property*:

$$\begin{aligned} \int_{\infty}^{-\infty} f(x)\delta(x-x_o)\, dx &= \int_{\infty}^{-\infty} f(x_o)\delta(x-x_o)\, dx \\ &= f(x_o) \int_{\infty}^{-\infty} \delta(x-x_o)\, dx \\ &= f(x_o) \end{aligned}$$

We can also show how a delta function can be used to shift a function:

$$\begin{aligned} f(x) * \delta(x-x_o) &= \int_{-\infty}^{\infty} f(\alpha)\delta(x-x_o-\alpha)\, d\alpha \\ &= \end{aligned}$$

Impulse response

We can discover the impulse response of an LSI system by feeding it a delta function:

$$g(x) = \delta(x) * h(x) = h(x)$$

Sampling

We can use the delta function to sample another function. As we indicated above:

$$f(x)\delta(x - x_o) = f(x_o)\delta(x - x_o)$$

To acquire a set of equi-spaced samples we can construct an impulse train:

$$\mathrm{III}(x) = \sum_{i=-\infty}^{\infty} \delta(x - i)$$

a.k.a., the *shah* or *comb* or *sampling function*.

To sample just multiply:

$$\begin{aligned} f(x)\mathrm{III}(x) &= f(x) \sum_{i=-\infty}^{\infty} \delta(x - i) \\ &= \sum_{i=-\infty}^{\infty} f(x)\delta(x - i) \\ &= \sum_{i=-\infty}^{\infty} f(i)\delta(x - i) \end{aligned}$$

Replication

If we convolve a function with the impulse train, we get many replicas of that function:

$$\begin{aligned} f(x) * \mathrm{III}(x) &= f(x) * \sum_{i=-\infty}^{\infty} \delta(x - i) \\ &= \sum_{i=-\infty}^{\infty} f(x) * \delta(x - i) \\ &= \sum_{i=-\infty}^{\infty} f(x - i) \end{aligned}$$

Fourier series

Consider a periodic function:

We can write this as a weighted sum of sines and cosines.

Let's consider just cosines for the even function above:

$$f(x) = \sum_{i=-\infty}^{\infty} a_i \cos 2\pi \frac{i}{T} x$$

We can compute a_i:

$$a_i = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left[2\pi \frac{i}{T} x\right] dx$$

Cosine transform

We can derive the cosine transform from the Fourier series of an even function.

First, plug the expressions for the Fourier series coefficients into the summation:

$$f(x) = \sum_{i=-\infty}^{\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left[2\pi \frac{i}{T} x\right] dx \cdot \cos 2\pi \frac{i}{T} x$$

Now define $\Delta s = \frac{1}{T}$:

$$\begin{aligned} f(x) &= \sum_{i=-\infty}^{\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left[2\pi i \Delta s x\right] dx \cdot \cos 2\pi i \Delta s x \cdot \Delta s \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \left\{ \sum_{i=-\infty}^{\infty} f(x) \cos\left[2\pi i \Delta s x\right] dx \right\} \cdot \cos 2\pi i \Delta s x \cdot \Delta s \end{aligned}$$

Cosine transform, cont'd

Now let $T \rightarrow \infty$:

Then:

$$\begin{aligned} \Delta s &\rightarrow \\ i\Delta s &\rightarrow \\ \Sigma &\rightarrow \end{aligned}$$

which leads to:

$$f(x) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x) \cos 2\pi s x \, dx \right\} \cos 2\pi s x \, ds$$

Cosine transform, cont'd

We can think of the term in brackets as the "cosine tranform" of $f(x)$:

$$a_i \rightarrow a(s) = \int_{-\infty}^{\infty} f(x) \cos 2\pi s x \, dx$$

And the "inverse cosine transform" would then be:

$$f(x) = \int_{-\infty}^{\infty} a(s) \cos 2\pi s x \, ds$$

Fourier transform

In general, functions must be represented as combinations of cosines and sines.

A compact way of "encoding" a sine and a cosine is given by the Euler relation:

$$e^{j\theta} = \cos\theta + j \sin\theta$$

This ultimately leads us to the *Fourier transform*:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi s x} \, dx$$

and the *inverse Fourier transform*:

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{j2\pi s x} \, dx$$

If $f(x)$ is even, then we get back exactly the cosine transform and its inverse.

Example: the "sinc" function

What is the Fourier transform of the rect function?

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sqcap(x) e^{-j2\pi s x}\, dx &= \int_{-1/2}^{1/2} e^{-j2\pi s x}\, dx \\
&= \frac{-1}{j2\pi s} e^{-j2\pi s x} \,\Big|_{-1/2}^{1/2} \\
&= \frac{-e^{-j\pi s x} + e^{j\pi s x}}{j2\pi s} \\
&= \frac{j2 \sin \pi s}{j2\pi s} \\
&= \frac{\sin \pi s}{\pi s} \\
&= \mathrm{sinc}(s)
\end{aligned}
$$

Useful theorems

Similarity theorem:

$$f(ax) \rightarrow \frac{1}{\mathrm{a}} F\left(\frac{s}{a}\right)$$

Shifting theorems:

$$
\begin{aligned}
f(x - x_o) &\rightarrow F(s) e^{-j2\pi x_o} \\
f(x) e^{j2\pi s_o} &\rightarrow F(s - s_o)
\end{aligned}
$$

Convolution theorems:

$$
\begin{aligned}
f * g &\rightarrow F \cdot G \\
f \cdot g &\rightarrow F * G
\end{aligned}
$$

Sampling and spectrum replication

The impulse train is the Fourier transform of itself:

$$\mathrm{III}(x) \rightarrow \mathrm{III}(s)$$

The impact of adjusting the sample spacings on the Fourier transform is:

$$\mathrm{III}(ax) \rightarrow \frac{1}{\mathrm{a}} \mathrm{III}\left(\frac{s}{a}\right)$$

Finally, sampling leads to spectrum replication:

$$f(x)\mathrm{III}(x) \rightarrow F(s)\mathrm{III}(s) = \sum_{i=-\infty}^{\infty} F(s - i)$$