Computer Graphics
CSE 557, Winter 1997

Week 3 study sheet: Curves and Surfaces

Problem 1 A Bézier curve $Q(u)$ of order n can be defined in terms of the locations of its $n+1$ control points $\{V_0, \ldots, V_n\}$:

$$Q(u) \quad = \quad \sum_{i=0}^{n} V_i \left(\begin{array}{c} n \\ i \end{array} \right) u^i \, (1-u)^{n-i}$$

True or false:

a) Every Bézier curve of any order interpolates the control points V_0 and V_n.

b) Every Bézier curve of any order is tangent to the line segments V_0V_1 and $V_{n-1}V_n$.

c) Every Bézier curve of any order lies inside the convex hull of its control points.

d) A Bézier curve of order 0 is just a point, and a Bézier curve of order 1 is just a line segment.

e) Every Bézier curve of order 3 lies in a plane.

Problem 1 (continued)

f) Suppose that you wrote a program to implement de Casteljau's algorithm directly, using C. What would be the asymptotic time complexity of computing each point $Q(t)$ using this algorithm, in terms of the degree n of the Bézier curve?

g) Use the de Casteljau algorithm to design a curve of degree 4 that has its middle control point on the curve. More specifically, try to achieve $V_2 = Q(\frac{1}{2})$.

h) Derive an analytic expression for V_2 in terms of the other control points that satisfies the (same) condition $V_2 = Q(\frac{1}{2})$.

Problem 1 (continued)

The points $\{V_0, V_1, V_2, V_3\}$ below are control points for a cubic Bézier curve $Q(t)$, for t varying on the interval $[0, 1]$. Recall that de Casteljau's algorithm can be used to construct the control points for the "left" and "right sub-curves," that is, the sub-curves corresponding to t varying on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively.

i) Construct, as carefully as you can on the diagram below, the Bézier control points $\{L_0, L_1, L_2, L_3\}$ and $\{R_0, R_1, R_2, R_3\}$ corresponding to the left and right sub-curves, respectively. Be sure to label these points on the diagram.

Problem 1 (continued)

The control points $\{L_0, L_1, L_2, L_3\}$ can also be computed algebraically through a matrix multiplication as follows:

$$\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{M} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

j) What are the entries of matrix M? (Hint: You should be able to derive these entries from the geometric constuction in part (i).)

Problem 1 (continued)

Since a Bézier curve $Q^n(u)$ can be expressed as a linear combination of control points

$$B_0(u)V_0 + B_1(u)V_1 + \cdots + B_n(u)V_n,$$

we can associate an interval $[u_{\min}, u_{\min}]$ with each control point V_i, defined as the region of the parameter domain u for which $B_i(u)$ is larger than any of the other "basis functions" $B_0(u), \ldots, B_n(u)$. We'll call this interval the *dominant region* of a control point V_i.

For example, for Bézier curves of degree 1, the dominant region of V_0 is $[0, 1/2]$, and the dominant region of V_1 is $[1/2, 1]$.

k) What is the dominant region for each of the three control points V_0, V_1, V_2 of a Bézier curve of degree 2?

Problem 2 More complex curves can be designed by piecing together different Bézier curves to make mathematical "splines." Two popular splines are the B-spline and the Catmull-Rom spline. If $\{B_0, B_1, B_2, B_3\}$ and $\{C_0, C_1, C_2, C_3\}$ are cubic B-spline and Catmull-Rom spline control points, respectively, then the corresponding Bézier control points $\{V_0, V_1, V_2, V_3\}$ can be constructed by the following identity:

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 0 & 6 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

True or false:

a) B-splines and Catmull-Rom splines both have C^2 continuity.

b) Neither B-splines nor Catmull-Rom splines interpolate their control points.

c) B-splines and Catmull-Rom splines both lie inside the convex hull of their control points.

d) B-splines and Catmull-Rom splines both provide local control.

Problem 2 (continued)

e) The points $\{B_0, B_1, B_2, B_3\}$ below are control points for a cubic B-spline. Construct, as carefully as you can on the diagram below, the Bézier control points $\{V_0, V_1, V_2, V_3\}$ corresponding to the same curve.

f) The points $\{C_0, C_1, C_2, C_3\}$ below are control points for a cubic Catmull-Rom spline. Construct, as carefully as you can on the diagram below, the Bézier control points $\{V_0, V_1, V_2, V_3\}$ corresponding to the same curve.

Problem 2 (continued)

One potential problem with cubic B-splines from a user's standpoint is that the curves do not actually begin or end very close to where the first and last control points B_0 and B_n were placed. One solution to this problem is to have the interactive system automatically place some extra "phantom" control points $\ldots, B_{-2}, B_{-1}$ and $B_{n+1}, B_{n+2}, \ldots$ at the beginning and end of the curve.

g) What is the effect on the curve if the first and last control points are "doubled"; that is, if an extra control point B_{-1} is placed at the beginning, and an extra control point B_{n+1} placed at the end of the sequence, directly on top of the first and last control points specified by the user?

h) What is the effect if the first and last control points are "tripled"?

i) What is the effect if a phantom control point B_{-1} is placed before the first control point of the curve, so that $B_0 - B_{-1} = B_1 - B_0$ (and similarly for the other end of the curve)?

j) The curves formed in methods (h) and (i) are similar in some respects. What is an advantage of using method (i) over method (h)?