

Affine transformations

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CSE 557
Autumn 2017

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Reading

Optional reading:

- ♦ Angel and Shreiner: 3.1, 3.7-3.11
- ♦ Marschner and Shirley: 2.3, 2.4.1-2.4.4, 6.1.1-6.1.4, 6.2.1, 6.3

Further reading:

- ♦ Angel, the rest of Chapter 3
- ♦ Foley, et al, Chapter 5.1-5.5.
- ♦ David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

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Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = f(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

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Vector representation

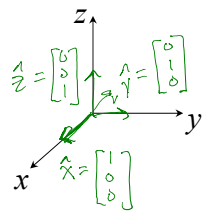
We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space:

- ♦ as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- ♦ as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$

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Canonical axes

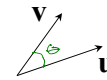


right-handed coord. systems

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$$



Vector length and dot products



$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$u^T v = \alpha$$

$$v^T u = \alpha^T = \alpha$$

$$u \cdot v \equiv u_x v_x + u_y v_y + u_z v_z = u^T v$$

$$u \cdot v = v \cdot u$$

$$(AB)^T = B^T A^T$$

$$v \cdot v = \|v\|^2$$

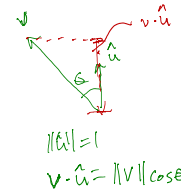
$$u \cdot v = \|u\| \|v\| \cos \theta$$

$$u \cdot v = 0 \iff \perp \text{ or orthogonal vectors}$$

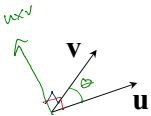
($\|u\|, \|v\| \neq 0$)

$$\|u\| = \|v\| = 1 \Rightarrow u \cdot v = \cos \theta$$

$$\hat{u} = \frac{u}{\|u\|} \Rightarrow \|\hat{u}\| = 1$$



Vector cross products



$$u \times v \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y) \hat{x} + (u_z v_x - u_x v_z) \hat{y} + (u_x v_y - u_y v_x) \hat{z}$$

if $u_z, v_z = 0$ $u \times v = u_x v_y - u_y v_x \hat{z}$

$$u \times v = -v \times u$$

$$(u \times v) \cdot u = 0$$

$$(u \times v) \cdot v = 0$$

$$\|u \times v\| = \|u\| \|v\| \sin \theta$$

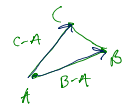
$$\hat{n} = \frac{u \times v}{\|u \times v\|}$$

$$\|u \times v\| = \|u\| \|v\| \sin \theta$$

$$\|u \times v\| = \text{Area}(\square_{u,v})$$

$$u \parallel v \iff \|u \times v\| = 0$$

$\|v\| \neq 0$



$$N = (B-A) \times (C-A)$$

$$\hat{N} = \frac{N}{\|N\|} \quad \text{Area}(\triangle ABC) = \frac{\|(B-A) \times (C-A)\|}{2}$$

$$\hat{x} \times \hat{y} = \hat{z}$$

for right-handed coord system

Representation, cont.

We can represent a 2-D transformation M by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If p is a column vector, M goes on the left:

$$p' = Mp$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If p is a row vector, M^T goes on the right:

$$p' = pM^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2×2 transformation matrix M :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$

$$y' = cx + dy$$

We will develop some intimacy with the elements $a, b, c, d \dots$

Identity

Suppose we choose $a = d = 1, b = c = 0$:

- Gives the **identity** matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Doesn't move the points at all

$$x' = x$$

$$y' = y$$

Scaling

Suppose we set $b = c = 0$, but let a and d take on any positive value:

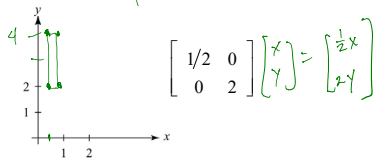
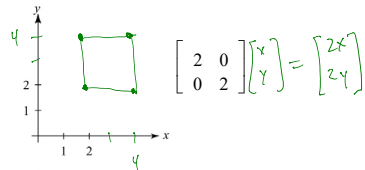
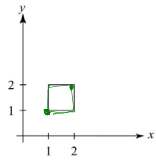
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in x and y :

$$x' = ax$$

$$y' = dy$$

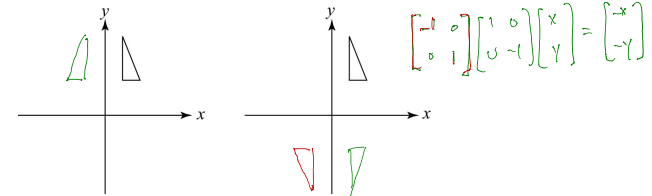


Reflection (mirror) transform

Suppose we keep $b = c = 0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



Shear

Now let's leave $a = d = 1$ and experiment with $b \dots$

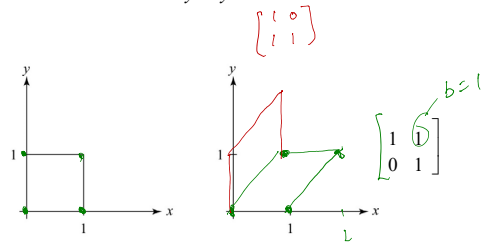
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives:

$$x' = x + by$$

$$y' = y$$



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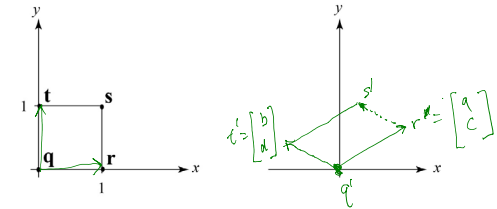
Effect on unit square

Let's see how a general 2×2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{q}' & \mathbf{r}' & \mathbf{s}' & \mathbf{t}' \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$

$\mathbf{q}' \quad \mathbf{r}' \quad \mathbf{s}' \quad \mathbf{t}'$



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Effect on unit square, cont.

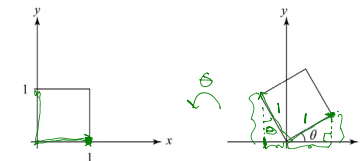
Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- a and d give x - and y -scaling
- b and c give x - and y -shearing

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Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Linear transformations

The unit square observations also tell us the 2x2 matrix transformation implies that we are representing a point in a new coordinate system:

$$\begin{aligned} \mathbf{p}' &= M\mathbf{p} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= x \cdot \mathbf{u} + y \cdot \mathbf{v} \end{aligned}$$

where $\mathbf{u}=[a \ c]^T$ and $\mathbf{v}=[b \ d]^T$ are vectors that define a new **basis** for a **linear space**.

The transformation to this new basis (a.k.a., change of basis) is a **linear transformation**.

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Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

translation

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Affine transformations

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space \mathbf{u}, \mathbf{v} with an origin \mathbf{t} .

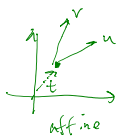
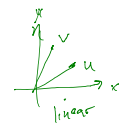
We call \mathbf{u}, \mathbf{v} , and \mathbf{t} (basis and origin) a **frame** for an **affine space**.

Then, we can represent a **change of frame** as:

$$\mathbf{p}' = x \cdot \mathbf{u} + y \cdot \mathbf{v} + \mathbf{t}$$

This change of frame is also known as an **affine transformation**.

How do we write an affine transformation with matrices?



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Homogeneous coordinates

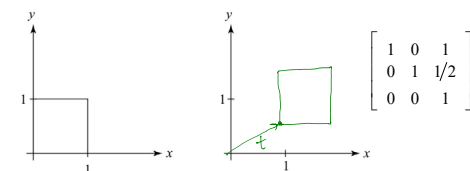
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "w" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+t_x \\ y+t_y \\ 1 \end{bmatrix}$$



... gives **translation!**

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Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} A & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by+t_x \\ cx+dy+t_y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{t} \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of $[0 \ 0 \ 1]$.

An "affine point" is a "linear point" with an added w -coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

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Rotation about arbitrary points

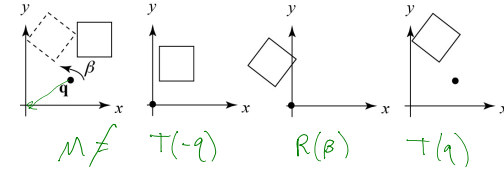
Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation by β , about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix.

Let's do this with rotation and translation matrices of the form $R(\theta)$ and $T(\mathbf{t})$, respectively.

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(\mathbf{t}) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



1. Translate \mathbf{q} to origin
2. Rotate
3. Translate back

$$M = T(\mathbf{q}) R(\beta) T(-\mathbf{q})$$

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Points and vectors

Vectors have an additional coordinate of $w = 0$. Thus, a change of origin has no effect on vectors.

$$\mathbf{B} - \mathbf{A} = \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} - \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} = \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix}$$

Q: What happens if we multiply a vector by an affine matrix?

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} av_x + bv_y \\ cv_x + dv_y \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector	→	vector	$\alpha \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha A_x + \beta B_x \\ \alpha A_y + \beta B_y \\ \alpha + \beta \end{bmatrix}$
scalar · vector	→	vector	
point - point	→	vector	$\alpha + \beta \geq 1 \Rightarrow$ point
point + vector	→	point	
point + point	→	chaos	$\alpha + \beta = 0 \Rightarrow$ vector
scalar · vector + scalar · vector	→	vector	
scalar · point + scalar · point	→	if depends	

One useful combination of affine operations is:

$$P(t) = P_0 + t\mathbf{u}$$

Q: What does this describe?

$$t \in (-\infty, \infty) \Rightarrow \text{line}$$

$$t \in [0, \infty) \Rightarrow \text{half-line ray}$$

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Combinations of points

Note that we seem to have constructed a point by adding points together, which we said was illegal, but as long as they have coefficients that sum to one, it's ok.

More generally, a weighted sum of points A_i :

$$P = \alpha_1 A_1 + \dots + \alpha_n A_n$$

is an **affine combination** if:

$$\sum_{i=1}^n \alpha_i = 1$$

It is a **vector combination** if:

$$\sum_{i=1}^n \alpha_i = 0$$

And it is a **convex combination** if:

$$\sum_{i=1}^n \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0$$

A convex combination of points will always lie in the convex hull of those points.

$$COM = \frac{\sum m_i x_i}{\sum m_j}$$

$$= \frac{\sum m_i x_i}{\sum m_j}$$

$$= \sum \frac{m_i}{\sum m_j} x_i$$

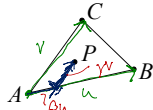
$$= \sum \alpha_i x_i$$

$$\alpha_i = \frac{m_i}{\sum m_j}$$

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Barycentric coordinates

A set of points can be used to create an affine frame.
Consider a triangle ABC and a point P :



We can form a frame with an origin C and the vectors from C to the other vertices:

$$\mathbf{u} = B - A \quad \mathbf{v} = C - A \quad \mathbf{t} = A$$

We can then write P in this affine coordinate frame:

$$\begin{aligned} P &= \beta \mathbf{u} + \gamma \mathbf{v} + \mathbf{t} \\ &= \beta(B-A) + \gamma(C-A) + A \\ &= \underbrace{(1-\beta-\gamma)}_{\alpha} A + \beta B + \gamma C \end{aligned}$$

The coordinates (α, β, γ) are called the **barycentric coordinates** of P relative to A, B , and C .

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Computing barycentric coordinates

Writing out the barycentric combination of points leads to a linear system:

$$\alpha A + \beta B + \gamma C = P \Rightarrow \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} \alpha + \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} \beta + \begin{bmatrix} C_x \\ C_y \\ 1 \end{bmatrix} \gamma = \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix}$$

Cramer's rule

$$\alpha = \frac{\begin{vmatrix} P_x & B_y & C_x \\ P_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{vmatrix}}$$

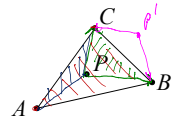
$$\alpha = \frac{\begin{vmatrix} P_x & B_x - P_x & C_x - P_x \\ P_y & B_y - P_y & C_y - P_y \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} A_x & B_x - P_x & C_x - P_x \\ A_y & B_y - P_y & C_y - P_y \\ 1 & 0 & 0 \end{vmatrix}} = \frac{(B-P) \times_{2D} (C-P)}{(B-A) \times_{2D} (C-A)}$$

A cool result is that we can write the solution as:

$$\alpha = \frac{(B-P) \times_{2D} (C-P)}{(B-A) \times_{2D} (C-A)} \quad \beta = \frac{(C-P) \times_{2D} (A-P)}{(B-A) \times_{2D} (C-A)} \quad \gamma = \frac{(A-P) \times_{2D} (B-P)}{(B-A) \times_{2D} (C-A)}$$

which can be interpreted geometrically as:

$$\alpha = \frac{\text{SArea}(PBC)}{\text{SArea}(ABC)} \quad \beta = \frac{\text{SArea}(APC)}{\text{SArea}(ABC)} \quad \gamma = \frac{\text{SArea}(ABP)}{\text{SArea}(ABC)}$$



Q: What does it mean for a barycentric coordinate to be negative? *P is outside $\triangle ABC$*

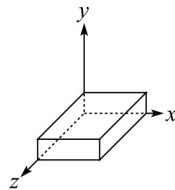
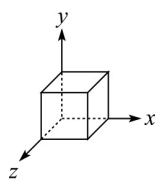
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Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

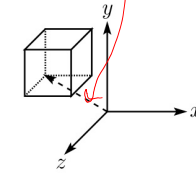
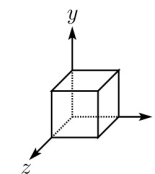
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Rotation in 3D

For any transformation, we can count its **degrees of freedom** – the number of independent (though not necessarily unique) parameters needed to specify the transformation.

One way to count them is to add up all the apparently free variables and subtract the number of independent equations that constrain them.

How many degrees of freedom are there in an arbitrary 3D rotation?

$R^T R = I$
 $R^{-1} = R^T$

$M = \begin{bmatrix} u & v & w \end{bmatrix}$ 9 variables

$u \cdot v = 0$
 $u \cdot w = 0$
 $v \cdot w = 0$

$u \cdot u = 1$
 $v \cdot v = 1$
 $w \cdot w = 1$

6 constraints

of DOFs = 9 vars. - 6 constr. = 3

$$R^T R = \begin{bmatrix} u^T & v^T & w^T \\ v^T & u^T & w^T \\ w^T & u^T & v^T \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

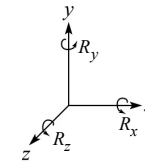
Rotation in 3D (cont'd)

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 & 0 \\ \sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

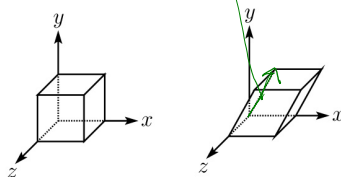
rotation around a vector (ϕ, \hat{v})

quaternion is a 4-vector that encodes ϕ, \hat{v}

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



We call this a shear with respect to the x-z plane.

Preservation of affine combinations

A transformation F is an affine transformation if it preserves affine combinations:

$$F(\alpha_1 A_1 + \dots + \alpha_n A_n) = \alpha_1 F(A_1) + \dots + \alpha_n F(A_n)$$

where the A_i are points, and:

$$\sum_{i=1}^n \alpha_i = 1$$

Clearly, the matrix form of F has this property.

One special example is a matrix that drops a dimension. For example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

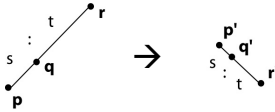
This transformation, known as an orthographic projection is an affine transformation.

We'll use this fact later...

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|pq\|}{\|qr\|} = \frac{s}{t} = \frac{\|p'q'\|}{\|q'r'\|}$$

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.