Mathematical surface representations

- Explicit \( z = f(x,y) \) (a.k.a., a “height field”)
  - what if the curve isn’t a function, like a sphere?

- Implicit \( g(x,y,z) = 0 \)

- Parametric \( S(u,v) = (x(u,v), y(u,v), z(u,v)) \)
  - For the sphere:
    \[ x(u,v) = r \cos 2\pi v \sin \pi u \]
    \[ y(u,v) = r \sin 2\pi v \sin \pi u \]
    \[ z(u,v) = r \cos \pi u \]

As with curves, we’ll focus on parametric surfaces.

Constructing surfaces of revolution

Given: A curve \( C(u) \) in the \( xy \)-plane:

\[
C(u) = \begin{bmatrix}
  x(u) \\
  y(u) \\
  z(u)
\end{bmatrix}
\]

Let \( R_y(\theta) \) be a rotation about the \( y \)-axis.

Find: A surface \( S(u,v) \) which is \( C(u) \) rotated about the \( y \)-axis, where \( u, v \in [0, 1] \).

Solution: \( S(u,v) = R_y(\pi v) C(u) \)
General sweep surfaces

The surface of revolution is a special case of a swept surface.

Idea: Trace out surface $S(u,v)$ by moving a profile curve $C(u)$ along a trajectory curve $T(v)$.

More specifically:
- Suppose that $C(u)$ lies in an $(x_c,y_c)$ coordinate system with origin $O_c$.
- For every point along $T(v)$, lay $C(u)$ so that $O_c$ coincides with $T(v)$.

Orientation

The big issue:
- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:
1. Fixed (or static): Just translate $O_c$ along $T(v)$.
2. Moving. Use the Frenet frame of $T(v)$.
   - Allows smoothly varying orientation.
   - Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.

To get a 3D coordinate system, we need 3 independent direction vectors.
- Tangent: $t(v) = \text{normalize}(T'(v))$
- Binormal: $b(v) = \text{normalize}(T'(v) \times T''(v))$
- Normal: $n(v) = b(v) \times t(v)$

As we move along $T(v)$, the Frenet frame $(t,b,n)$ varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:
- Put $C(u)$ in the normal plane.
- Place $O_c$ on $T(v)$.
- Align $x_c$ for $C(u)$ with $b$.
- Align $y_c$ for $C(u)$ with $n$.

If $T(v)$ is a circle, you get a surface of revolution exactly!
Degenerate frames

Let’s look back at where we computed the coordinate frames from curve derivatives:

Where might these frames be ambiguous or undetermined?

Variations

Several variations are possible:
- Scale \( C(u) \) as it moves, possibly using length of \( T(v) \) as a scale factor.
- Morph \( C(u) \) into some other curve \( \tilde{C}(u) \) as it moves along \( T(v) \).
- ...

Tensor product Bézier surfaces

Given a grid of control points \( V_{ij} \) forming a control net, construct a surface \( S(u,v) \) by:
- treating rows of \( V \) (the matrix consisting of the \( V_{ij} \)) as control points for curves \( V_{10}(u), \ldots, V_{n0}(u) \).
- treating \( V_{10}(u), \ldots, V_{n0}(u) \) as control points for a curve parameterized by \( v \).

Tensor product Bézier surfaces, cont.

Let’s walk through the steps:

Which control points are interpolated by the surface?
Polyomial form of Bézier surfaces

Recall that cubic Bézier curves can be written in terms of the Bernstein polynomials:

\[ Q(u) = \sum_{i=0}^{3} V_i b_i(u) \]

A tensor product Bézier surface can be written as:

\[ S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} V_{ij} b_i(u)b_j(v) \]

In the previous slide, we constructed curves along \( u \), and then along \( v \). This corresponds to re-grouping the terms like so:

\[ S(u,v) = \sum_{i=0}^{n} \left( \sum_{j=0}^{m} V_{ij} b_j(v) \right) b_i(u) \]

But, we could have constructed them along \( v \), then \( u \):

\[ S(u,v) = \sum_{j=0}^{m} \left( \sum_{i=0}^{n} V_{ij} b_i(u) \right) b_j(v) \]

Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce \( C^2 \) continuity and local control, we get B-spline curves:

- treat rows of \( B \) as control points to generate Bézier control points in \( u \).
- treat Bézier control points in \( u \) as B-spline control points in \( v \).
- treat B-spline control points in \( v \) to generate Bézier control points in \( u \).

Tensor product B-spline surfaces, cont.

Another example:

Which B-spline control points are interpolated by the surface?
NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:

We can do this by trimming the \( u \)\( -v \) domain.

- Define a closed curve in the \( u \)\( -v \) domain (a trim curve)
- Do not draw the surface points inside of this curve.

It’s really hard to maintain continuity in these regions, especially while animating.