Fourier analysis and sampling theory

Reading

Required:

- Shirley, Ch. 9

Recommended:

- Ron Bracewell, The Fourier Transform and Its Applications, McGraw-Hill.
- Don P. Mitchell and Arun N. Netravali, "Reconstruction Filters in Computer Computer Graphics," Computer Graphics, (Proceedings of SIGGRAPH 88). *22 (4)*, pp. 221-228, 1988.

What is an image?

We can think of an **image** as a function, f, from R^2 to R:

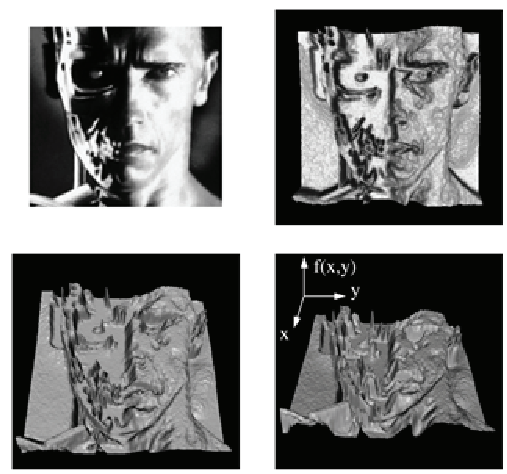
- $f(x,y)$ gives the intensity of a channel at position (x,y)
- Realistically, we expect the image only to be defined over a rectangle, with a finite range:
  - $f: [a,b] \mathrm{x} [c,d] \rightarrow [0,1]$

A color image is just three functions pasted together. We can write this as a "vector-valued" function:

$$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

We'll focus in grayscale (scalar-valued) images for now.

Images as functions



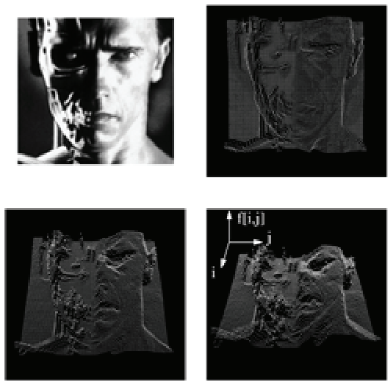
Digital images

In computer graphics, we usually create or operate on **digital** (**discrete**) images:

- **Sample** the space on a regular grid
- **Quantize** each sample (round to nearest integer)

If our samples are Δ apart, we can write this as:

$$f[n,m] = \text{Quantize}\{\, f(n\Delta, m\Delta)\,\}$$



Motivation: filtering and resizing

What if we now want to:

- smooth an image?
- sharpen an image?
- enlarge an image?
- shrink an image?

In this lecture, we will explore the mathematical underpinnings of these operations.

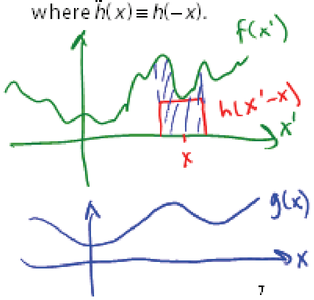
Convolution

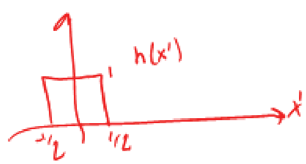
One of the most common methods for filtering a function, e.g., for smoothing or sharpening, is called **convolution**.

In 1D, convolution is defined as:

$$\begin{aligned} g(x) &= f(x) * h(x) \\ &= \int_{-\infty}^{\infty} f(x')h(x-x')dx' \\ &= \int_{-\infty}^{\infty} f(x')\tilde{h}(x'-x)dx' \end{aligned}$$

where $\tilde{h}(x) \equiv h(-x)$.



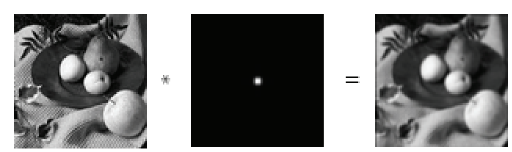


Convolution in 2D

In two dimensions, convolution becomes:

$$\begin{aligned} g(x,y) &= f(x,y) * h(x,y) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy' \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\tilde{h}(x'-x,y'-y)dx'dy' \end{aligned}$$

where $\tilde{h}(x,y) = h(-x,-y)$.



$f(x,y)$ $h(x,y)$ $g(x,y)$

Fourier transforms

Convolution, while a bit cumbersome looking, actually has a beautiful structure when viewed in terms of **Fourier analysis**.

We can represent functions as a weighted sum of sines and cosines.

We can think of a function in two complementary ways:

- **Spatially** in the **spatial domain**
- **Spectrally** in the **frequency domain**

The **Fourier transform** and its inverse convert between these two domains:

$$\text{Spatial domain} \longrightarrow F(s)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi sx}dx \longrightarrow \text{Frequency domain}$$

$$\text{Spatial domain} \longleftarrow f(x)=\int_{-\infty}^{\infty} F(s)e^{i2\pi sx}ds \longleftarrow \text{Frequency domain}$$

Fourier transforms (cont'd)

$$\text{Spatial domain} \longrightarrow F(s)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi sx}dx \longrightarrow \text{Frequency domain}$$

$$\text{Spatial domain} \longleftarrow f(x)=\int_{-\infty}^{\infty} F(s)e^{i2\pi sx}ds \longleftarrow \text{Frequency domain}$$

$f(x)$ is usually a real signal, but $F(s)$ is generally complex:

$$F(s)=A(s)+iB(s)=|F(s)|e^{i2\pi\theta(s)}$$

where magnitude $|F(s)|$ and phase $\theta(s)$ are:

$$|F(s)|=\sqrt{A^2(s)+B^2(s)}$$

$$\theta(s)=\tan^{-1}[B(s)/A(s)]$$

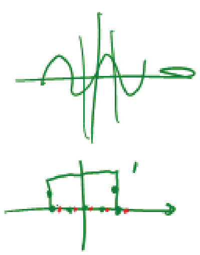
Where do the sines and cosines come in?

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

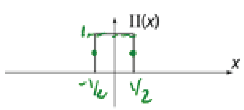
$$e^{-i2\pi sx} = \cos 2\pi sx - i\sin(2\pi sx)$$

Fourier transform example

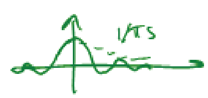
$\text{sinc}(x) \longleftrightarrow \Pi(s)$



Spatial domain



Frequency domain



$$\Pi(x)=\begin{cases}1 & |x|<1/2\\ 1/2 & |x|=1/2\\ 0 & |x|>1/2\end{cases} \longleftrightarrow \text{sinc}(s)$$

$\text{sinc}(s)$

$$\frac{\pi\cos\pi s}{\pi}$$

$$F(s) = \int_{-\infty}^{\infty}\Pi(x)e^{-i2\pi sx}dx$$

$$= \int_{-\infty}^{\infty}\Pi(x)\cos 2\pi sx\,dx - i\int_{-\infty}^{\infty}\Pi(x)\sin 2\pi sx\,dx$$

$$= \int_{-1/2}^{1/2}\cos 2\pi sx\,dx - i\cancel{\int_{-1/2}^{1/2}\sin 2\pi sx\,dx}$$

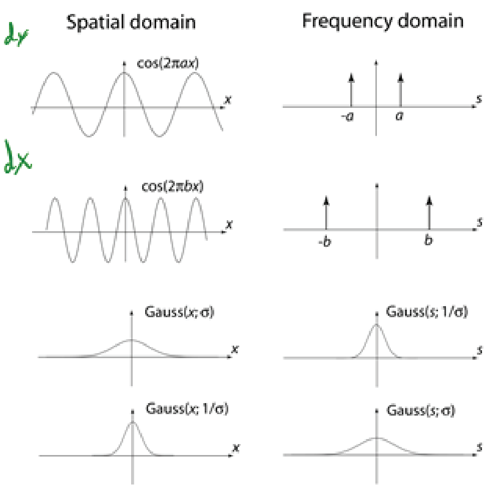
$$= \frac{\sin 2\pi sx}{2\pi s}\Bigg|_{-1/2}^{1/2} = \frac{\sin\pi s}{2\pi s} - \frac{-\sin\pi s}{2\pi s} = \frac{2\sin\pi s}{2\pi s}$$

$$= \frac{\sin\pi s}{\pi s}$$

More 1D Fourier examples

$$\int\cos 2\pi ax\cos 2\pi sx\,dx$$

$$=\frac{1}{2}\int\cos 2\pi(ax-sx) + \cos 2\pi(ax+sx)\,dx$$



Some properties of FT's

$\mathrm{var}\,x \cdot \mathrm{var}\,p \geq \hbar$

Heisenberg uncertainty princ.

$$F(s)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi sx}dx$$

Amplitude scaling:

$g(x)=kf(x)$

$\mathcal{F}\{g(x)\}=\int_{-\infty}^{\infty} kf(x)e^{-i\cdots}dx = k\int_{-\infty}^{\infty} f(x)e^{-i\cdots}dx = kF(s)$

Additivity:

$f(x)+g(x) \longleftrightarrow F(s)+G(s)$

Domain scaling:

$g(x)=f(ax)$

$\mathcal{F}\{g(x)\}=\int_{-\infty}^{\infty} f(ax)e^{-i2\pi sx}dx$

$x'=ax$, $dx'=a\,dx \to dx=\frac{1}{a}dx'$, $x=\frac{1}{a}x'$

$=\int_{-\infty}^{\infty} f(x')e^{-i2\pi\frac{s}{a}x'}\cdot\frac{1}{a}dx' = \frac{1}{a}\int_{-\infty}^{\infty} f(x')e^{-i2\pi\frac{s}{a}x'}dx'$

$=\frac{1}{|a|}F\left(\frac{s}{a}\right)$

2D Fourier transform

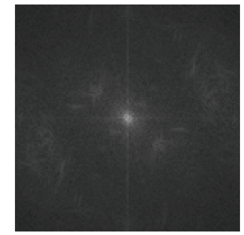
Spatial domain $\longrightarrow$ $F(s_x,s_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(s_x x+s_y y)}dxdy$ $\longrightarrow$ Frequency domain

Frequency domain $\longrightarrow$ $f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s_x,s_y)e^{i2\pi(s_x x+s_y y)}ds_x ds_y$ $\longrightarrow$ Spatial domain

Spatial domain

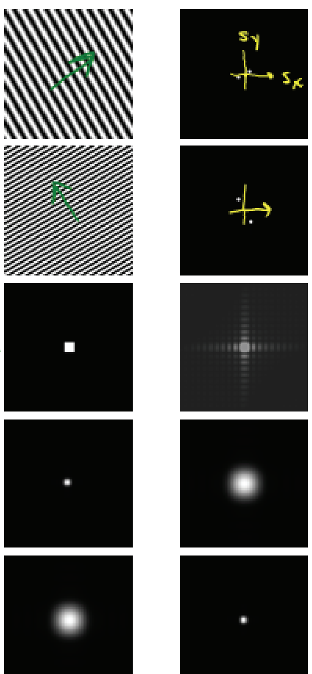
$f(x,y)$

Frequency domain



$|F(s_x,s_y)|$

2D Fourier examples



Spatial domain

$f(x,y)$

$\Pi(x)\Pi(y)$

Frequency domain

$|F(s_x,s_y)|$

$\iint \Pi(x)\Pi(y)e^{-i2\pi(s_x x+s_y y)}dxdy$

$=\underbrace{\int \Pi(x)e^{-i2\pi s_x x}dx}\underbrace{\int \Pi(y)e^{-i2\pi s_y y}dy}$

$=\mathrm{sinc}(s_x)\,\mathrm{sinc}(s_y)$

Fourier transforms and convolution

What is the Fourier transform of the convolution of two functions? (The answer is very cool!)

$$f*h \longleftrightarrow ??$$

$\mathcal{F}\{f*h\}=\int_{-\infty}^{\infty}\int f(x')h(x-x')dx'\,e^{-i2\pi sx}dx$

$=\int_{-\infty}^{\infty} f(x')\int_{-\infty}^{\infty} h(x-x')e^{-i2\pi sx}dx\,dx'$

$x''=x-x'$ $\quad \frac{dx''}{dx}=1 \Rightarrow dx''=dx$ $\quad x=x''+x'$

$=\int_{-\infty}^{\infty} f(x')\int_{-\infty}^{\infty} h(x'')e^{-i2\pi s(x''+x')}dx''dx'$

$=\underbrace{\int_{-\infty}^{\infty} f(x')e^{-i2\pi sx'}dx'}\underbrace{\int_{-\infty}^{\infty} h(x'')e^{-i2\pi sx''}dx''}$

$=F(s)H(s)$

Convolution theorems

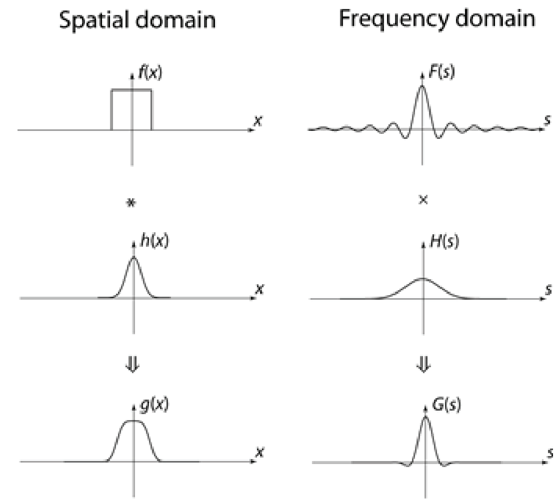
Convolution theorem: *Convolution* in the *spatial* domain is equivalent to *multiplication* in the *frequency* domain.

$$f * h \longleftrightarrow F \cdot H$$

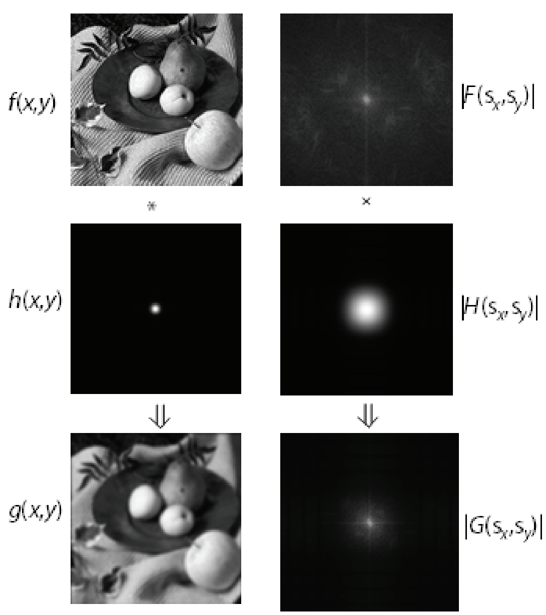
Symmetric theorem: *Convolution* in the *frequency* domain is equivalent to *multiplication* in the *spatial* domain.

$$f \cdot h \longleftrightarrow F * H$$

1D convolution theorem example



2D convolution theorem example



Convolution properties

Convolution exhibits a number of basic, but important properties…easily proved in the Fourier domain.

Commutativity:

$$a(x) * b(x) = b(x) * a(x)$$

$$A(s)\,B(s) = B(s)\,A(s)$$

Associativity:

$$[a(x) * b(x)] * c(x) = a(x) * [b(x) * c(x)]$$

$$[A(s)\,B(s)]\,C(s) = A(s)\,[B(s)\,C(s)]$$

Linearity:

$$a(x) * [k \cdot b(x)] = k \cdot [a(x) * b(x)]$$

$$A(s) \cdot K B(s) = K[A(s) \cdot B(s)]$$

$$a(x) * (b(x) + c(x)) = a(x) * b(x) + a(x) * c(x)$$

$$A(s) \cdot (B(s) + C(s)) = A(s)\,B(s) + A(s)\,B(s)$$

The delta function

$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx}\,dx$

$F(0) = \int_{-\infty}^{\infty} f(x)\,dx$

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ Kronecker Delta

The **Dirac delta function** (or **impulse function**), $\delta(x)$, is a handy tool for sampling theory.

It has zero width, infinite height, and unit area.

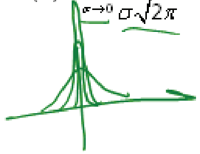
Can be computed as a limit of various functions, e.g.:

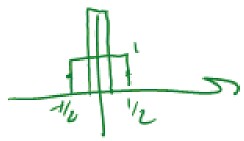
$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) = \lim_{w \to 0} \frac{1}{w} \Pi\left(\frac{x}{w}\right)$$

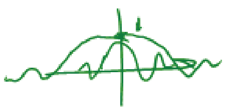
$\mathcal{F}\{\delta(x)\} = 1$

$\Pi(x) \rightarrow \text{sinc}(s)$

$\frac{1}{w}\Pi\left(\frac{x}{w}\right) \rightarrow \text{sinc}(ws)$



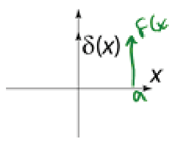




It is usually drawn as:

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$



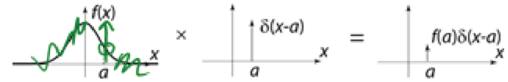
Sifting and shifting

$f(x)\delta(x) = f(0)\delta(x)$

For sampling, the delta function has two important properties.

Sifting:

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

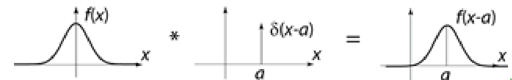


Shifting:

$$f(x) * \delta(x-a) = f(x-a)$$

$f(x) * \delta(x) = f(x)$

$F(s) \cdot 1 = F(s)$



$f(x) * \delta(x-a) = \int \delta(x'-a) f(x-x')\,dx'$

$= \int \delta(x'-a) f(x-a)\,dx'$

$= f(x-a) \int \delta(x'-a)\,dx'$

$= f(x-a)$

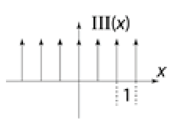
The shah/comb function

A string of delta functions is the key to sampling. The resulting function is called the **shah** or **comb** function or **impulse train**:

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

Ш

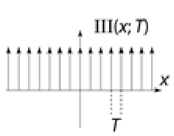
which looks like:



We can also define an impulse train in terms of a desired delta function spacing, T:

$$\text{III}(x;T) = \sum_{n=-\infty}^{\infty} \delta(x-nT)$$

which looks like:



The shah/comb function, cont'd

If we multiply an input function by the impulse train, we get:

Sifting

$$f(x)\text{III}(x;T) = f(x)\sum_{n=-\infty}^{\infty} \delta(x-nT)$$

$= \sum f(x)\delta(x-nT)$

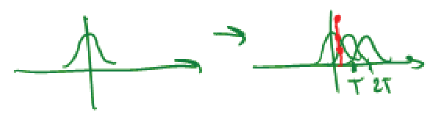
$= \sum f(nT)\delta(x-nT)$

Shifting

$f(x) * \text{III}(x;T) = f(x) * \sum \delta(x-nT)$

$= \sum f(x) * \delta(x-nT)$

$= \sum f(x-nT)$



The shah/comb function, cont'd

Amazingly, the Fourier transform of the shah function is also the shah function:

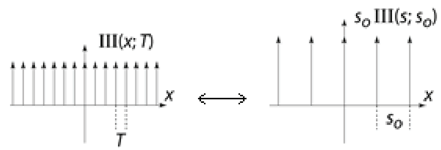
$$\text{III}(x) \longleftrightarrow \text{III}(s)$$

One can also show that:

$$\text{III}(x;T) \longleftrightarrow \frac{1}{T}\text{III}(s;1/T) = s_o \text{III}(s;s_o)$$

where $s_o = 1/T$.

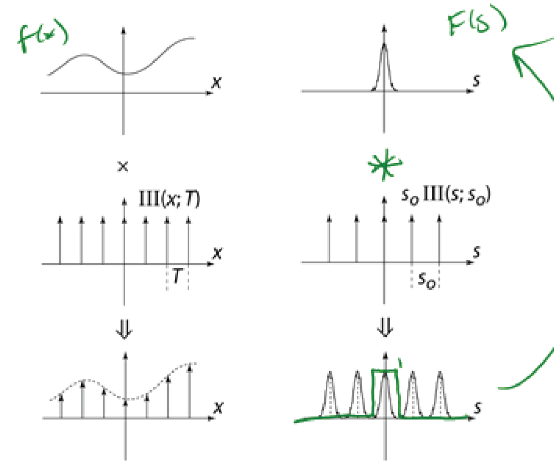
We can visualize this as:



For convenience, I won't draw the delta functions as scaled vertically, though mathematically, one must keep track of these scale factors.

Sampling

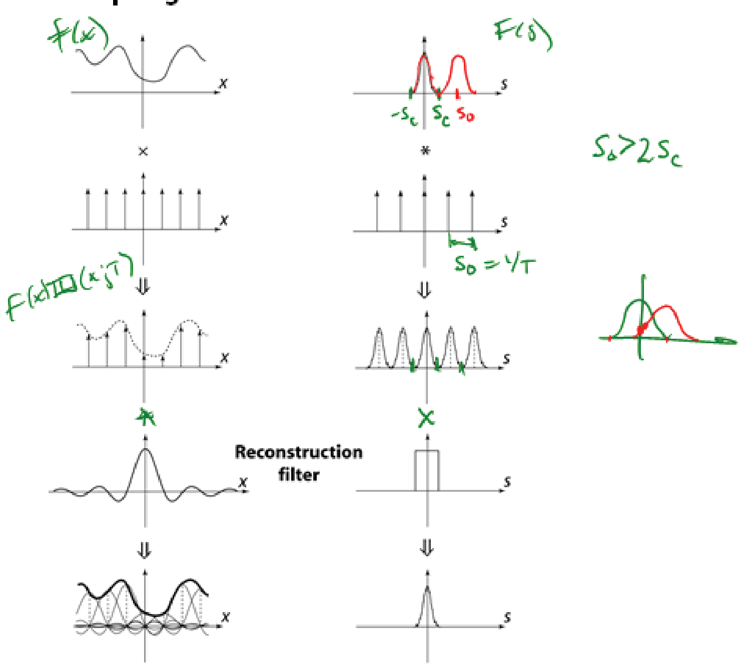
Now, we can talk about sampling.



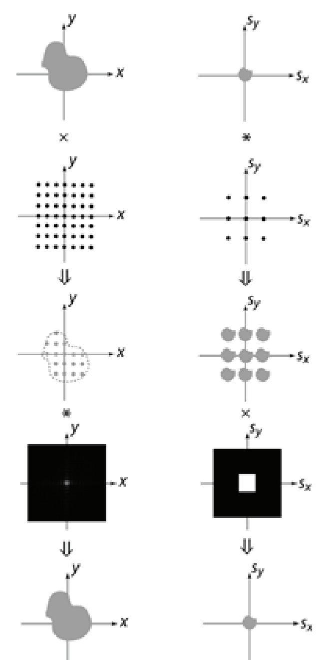
The Fourier spectrum gets *replicated* by spatial sampling!

How do we recover the signal?

Sampling and reconstruction



Sampling and reconstruction in 2D



Sampling theorem

This result is known as the **Sampling Theorem** and is due to Claude Shannon who first discovered it in 1949:

> A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above ½ the sampling frequency.

$S_o > 2S_c$

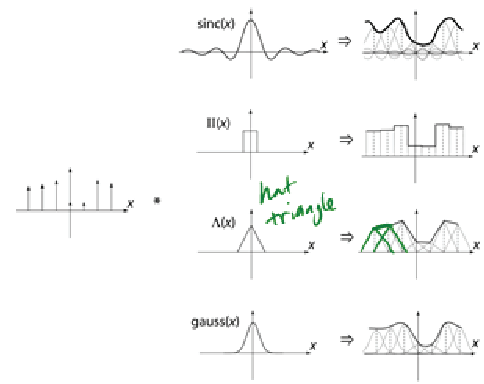
For a given **band limited** function, the minimum rate at which it must be sampled is the **Nyquist frequency**.

Reconstruction filters

The sinc filter, while "ideal", has two drawbacks:

- It has large support (slow to compute)
- It introduces ringing in practice

We can choose from many other filters…



nearest neighbor interpolation

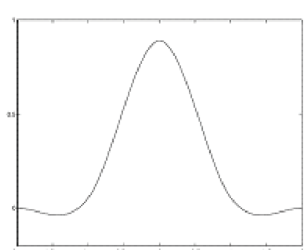
Cubic filters

Mitchell and Netravali (1988) experimented with cubic filters, reducing them all to the following form:

$$r(x)=\frac{1}{6}\begin{cases}(12-9B-6C)|x|^3+(-18+12B+6C)|x|^2+(6-2B) & |x|<1\\ ((-B-6C)|x|^3+(6B+30C)|x|^2+(-12B-48C)|x|+(8B+24C) & 1\le|x|<2\\ 0 & \textit{otherwise}\end{cases}$$

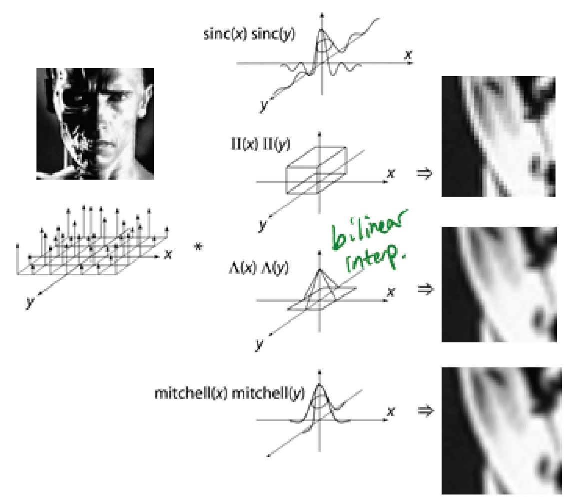
The choice of B or C trades off between being too blurry or having too much ringing. B=C=1/3 was their "visually best" choice.

The resulting reconstruction filter is often called the "Mitchell filter."



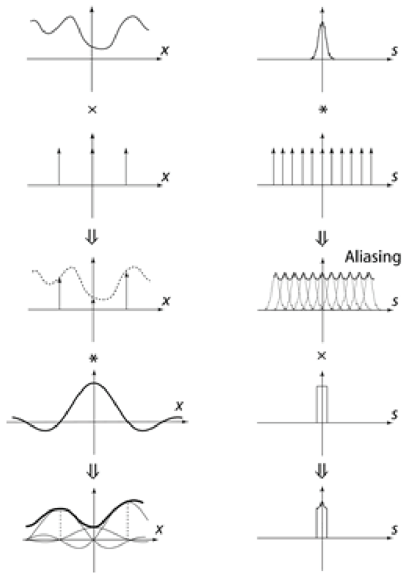
Reconstruction filters in 2D

We can also perform reconstruction in 2D…



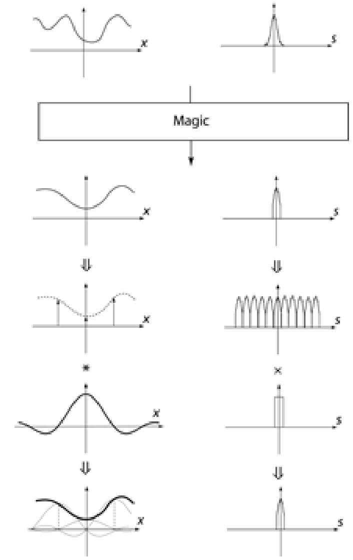
Aliasing

What if we go below the Nyquist frequen cy?



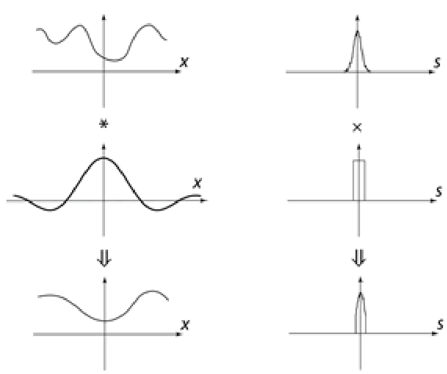
Anti-aliasing

Anti-aliasing is the process of *removing* the frequen cies before they alias.



Anti-aliasing by analytic prefiltering

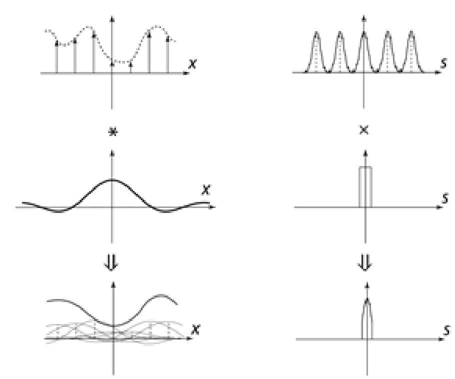
We can fill the "magic" box with analytic pre-filtering of the signal:



Why may this not generally be possible?

Filtered downsampling

Alternatively, we can sample the image at a higher rate, and then filter that signal:



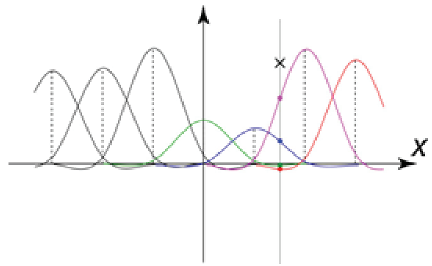
We can now sample the signal at a lower rate. The whole process is called **filtered downsampling** or **supersampling and averaging down**.

Practical upsampling

When resampling a function (e.g., when resizing an image), you do not need to reconstruct the complete continuous function.

For zooming in on a function, you need only use a reconstruction filter and evaluate as needed for each new sample.

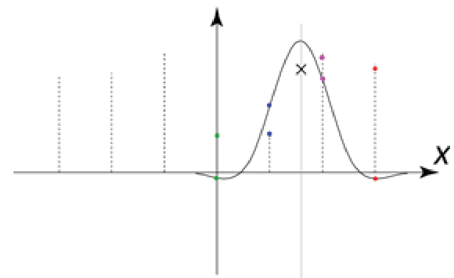
Here's an example using a cubic filter:



Practical upsampling

This can also be viewed as:

1. putting the reconstruction filter at the desired location
2. evaluating at the original sample positions
3. taking products with the sample values themselves
4. summing it up



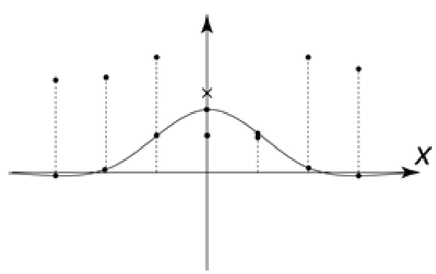
Important: filter should always be normalized!

Practical downsampling

Downsampling is similar, but filter has larger support and smaller amplitude.

Operationally:

1. Choose reconstruction filter in downsampled space.
2. Compute the downsampling rate, d, ratio of new sampling rate to old sampling rate
3. Stretch the filter by $1/d$ and scale it down by d
4. Follow upsampling procedure (previous slides) to compute new values



2D resampling

We've been looking at **separable** filters:

$$r_{2D}(x, y) = r_{1D}(x) r_{1D}(y)$$

How might you use this fact for efficient resampling in 2D?