Affine Transformations

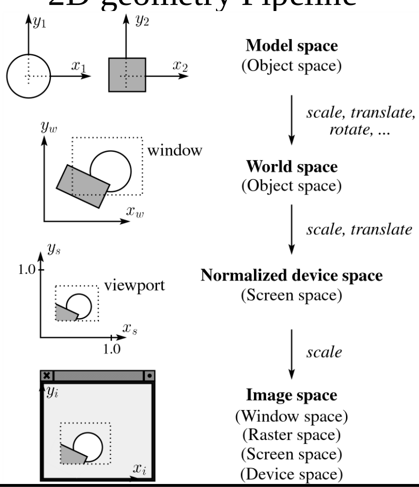
Reading

- Foley et al., Chapter 5.6 and Chapter 6

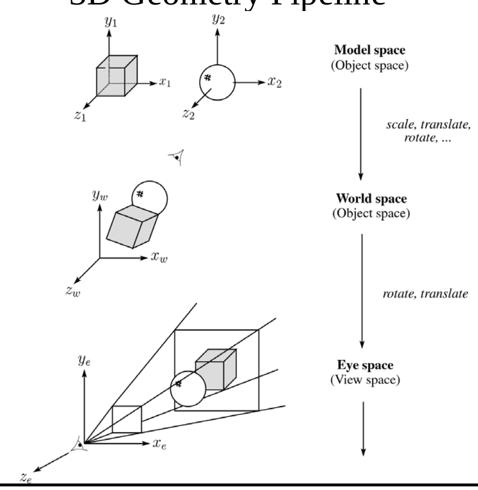
Supplemental

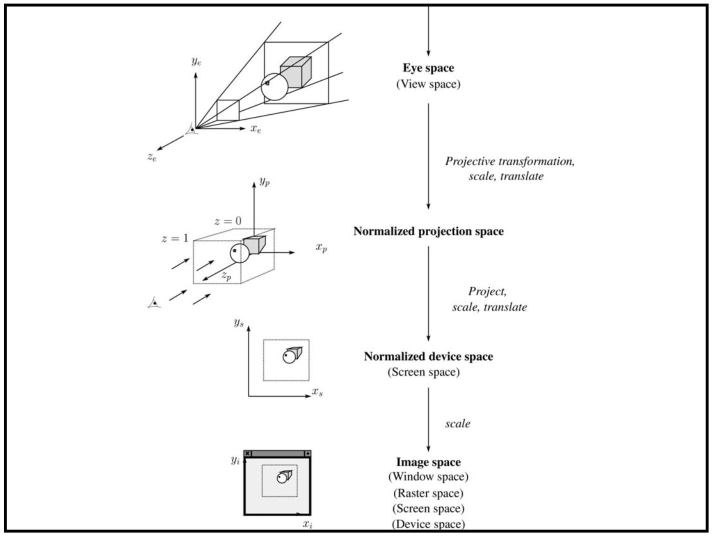
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics, Second edition*

2D geometry Pipeline



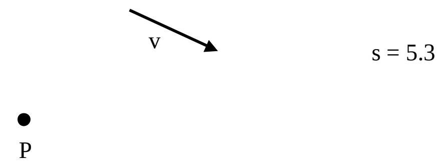
3D Geometry Pipeline





Affine Geometry

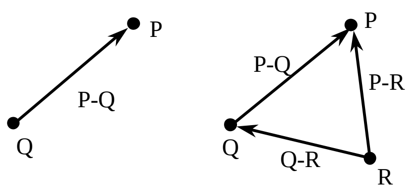
- Points: location in 3D space
- Vectors: quantity with a direction and magnitude, but no fixed position
- Scalar: a real number



Affine Spaces

Affine space consists of points and vectors related by a set of axioms:

- Difference of two points is a vector:
- Head-to-tail rule for vector addition:



Affine Operations

Legal affine operations:

$$\text{vector} + \text{vector} \rightarrow \text{vector}$$
$$\text{scalar} \cdot \text{vector} \rightarrow \text{vector}$$
$$\text{point} - \text{point} \rightarrow \text{vector}$$
$$\text{point} + \text{vector} \rightarrow \text{point}$$

… example of an "illegal" operation:

$$\text{point} + \text{point} \rightarrow \text{nonsense}$$

Useful combination of affine operations:

$$P(\alpha) = P_0 + \alpha \mathbf{v}$$

What is it?

Affine Combination

Affine combination of two points:

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2$$

where $\alpha_1 + \alpha_2 = 1$ is defined to be the point

$$Q = Q_1 + \alpha_1 (Q_2 - Q_1)$$

We can generalize affine combination to multiple points:

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_n Q_n$$

where

$$\sum \alpha_i = 1$$

Affine Frame

A frame can be defined as a set of vectors and a point:

$$(\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathcal{O})$$

Where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ form a basis and $\mathcal{O}$ is a point in space.

Any point P can be written as

$$P = p_1 \mathbf{v}_1 + \cdots + p_n \mathbf{v}_n + \mathcal{O}$$

And any vector as:

$$\mathbf{u} = u_1 \mathbf{v}_1 + \cdots + u_n \mathbf{v}_n$$

Matrix representation of points and vectors

Coordinate axiom: $0 \cdot P = \mathbf{0}$

$$\cdot P = P$$

So every point in the frame $\Phi = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathcal{O})$ can be written as

$$P = p_1 \mathbf{v}_1 + p_2 \mathbf{v}_2 + \cdots + p_n \mathbf{v}_n + 1 \cdot \mathcal{O}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \\ 1 \end{bmatrix}$$

And every vector as

$$\mathbf{u} = u_1 \mathbf{v}_1 + u_2 \mathbf{v}_2 + \cdots + u_n \mathbf{v}_n + 0 \cdot \mathcal{O}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \\ 0 \end{bmatrix}$$

Changing frames

Given a point P in frame Φ, what are the coordinates of P in frame $\Phi' = (\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_n', \mathcal{O}')$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathcal{O} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1' & \mathbf{v}_2' & \cdots & \mathbf{v}_n' & \mathcal{O}' \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \\ \cdots \\ p_n' \\ 1 \end{bmatrix}$$

Since each element of Φ can be written in coordinates relative to Φ'

$$\mathbf{v}_i = f_{i,1} \mathbf{v}_1' + \cdots + f_{i,n} \mathbf{v}_n'$$
$$\mathcal{O} = f_{n+1,1} \mathbf{v}_1' + \cdots + f_{n+1,n} \mathbf{v}_n' + \mathcal{O}'$$

Changing frames cont'd

Written in a matrix form

$$[\mathbf{v}_1' \quad \mathbf{v}_2' \quad \cdots \quad \mathbf{v}_n' \quad O'] \begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \\ 1 \end{bmatrix} = [\mathbf{v}_1' \quad \mathbf{v}_2' \quad \cdots \quad \mathbf{v}_n' \quad O'] \begin{bmatrix} f_{1,1} & \cdots & f_{n,1} & f_{n+1,1} \\ \vdots & \ddots & & \vdots \\ f_{1,n} & & f_{n,n} & f_{n+1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \\ 1 \end{bmatrix} = \begin{bmatrix} f_{1,1} & \cdots & f_{n,1} & f_{n+1,1} \\ \vdots & \ddots & & \vdots \\ f_{1,n} & & f_{n,n} & f_{n+1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix} = \mathbf{F} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ 1 \end{bmatrix}$$

Euclidean and Cartesian spaces

A Euclidean space is an affine space with an inner product:

$$\langle u, v \rangle = u \cdot v = u^T v$$

A Cartesian space is a Euclidean space with a standard orthonormal frame. In 3D: (**i**, **j** , **k**, O)

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Useful properties and operations in Cartesian spaces

Length: $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Distance between points: $|P - Q|$

Angle between vectors: $\cos^{-1}\left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} \right)$

Perpendicular (orthogonal): $\mathbf{u} \cdot \mathbf{v} = 0$

Parallel: $\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \pm 1$

Cross product (in 3D): $\mathbf{u} \times \mathbf{v} = \mathbf{w}$

Affine Transformations

$F : A \to B$ is an affine transformation if it preserves affine combinations:

$$F\left(\sum \alpha_i Q_i \right) = \sum \alpha_i F(Q_i)$$

Where $\sum \alpha_i = 1$. The same applies to vectors.

Affine coordiantes are preserved: $F\left(\mathrm{O} + \sum p_i \mathbf{v}_i \right) = F(\mathrm{O}) + \sum p_i F(\mathbf{v}_i)$

Lines map to lines: $F(P_0 + \alpha \mathbf{v}) = F(P_0) + \alpha F(\mathbf{v})$

Paralelism is preserved: $F(Q_0 + \beta \mathbf{v}) = F(Q_0) + \beta F(\mathbf{v})$

Ratios are preserved: $Ratio\left(Q_1, Q, Q_2\right) = Ratio\left(F(Q_1), F(Q), F(Q_2)\right)$

2D Affine Transformations

P=[x,y,1]

P is a column vector $P' = \mathbf{M}P$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P is a row vector

$$P' = P\mathbf{M}$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 1 \end{bmatrix}$$

Identity

Doesn't move points at all

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + c$$

$$y' = y + f$$

Scaling

Changing the diagonal elements performs scaling

$$\begin{bmatrix} a & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x' &= ax \\ y' &= fy \end{aligned}$$

If $a=f$ scaling is uniform

What if $a,f<0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing

What about the off-diagonal elements?

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Gives

$$x' = x$$

$$y' = dx + y$$

Effect on unit square

$$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & d & d+e & e \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- M can be determined just by knowing how corners [1,0,1] and [0,1,1] are mapped
- A and f give x- and y-scaling
- B and d give x- and y-shearing

Rotation

- Rotation of points [1,0,1] and [0,1,1] by angle α around the origin:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \\ 1 \end{bmatrix}$$

The Matrices

Identity (do nothing): $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scale by s_x in the x and s_y in the y direction ($s_x < 0$ or $s_y < 0$ is reflection): $\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotate by angle θ (in radians): $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Shear by amount a in the x direction: $\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Shear by amount b in the y direction: $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Translate by the vector (t_x, t_y): $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Transformation Composition

Applying transformations **F** to point P and transformation **G** to the result

$$P' = \mathbf{F}P$$

$$P'' = \mathbf{G}P'$$

Combining two transformations

$$\begin{aligned} P'' &= \mathbf{G}(\mathbf{F}P) \\ &= (\mathbf{GF})P \end{aligned}$$

Rotation around arbitrary point

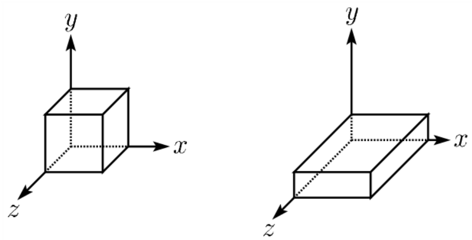
Reflection around arbitrary axis

Properties of Transforms

- Compact representation
- Fast implementation
- Easy to invert
- Easy to compose

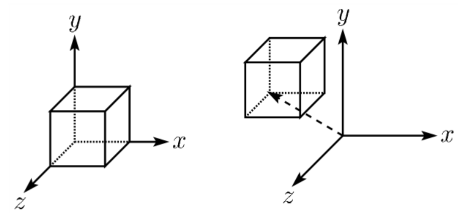
3D Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & t_x \\ 0 & 0 & 0 & t_y \\ 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation

Rotate by x-axis
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate by y-axis
$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate by z-axis
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How can we rotate about an *arbitrary line*?

3D Shear

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$