

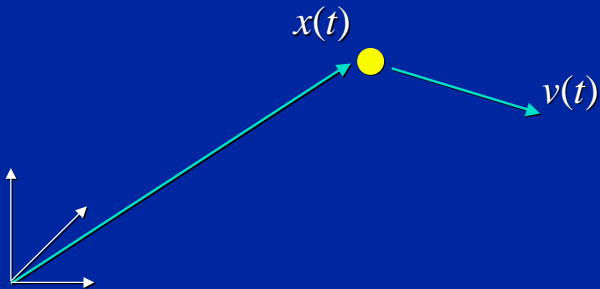
# Rigid Body Simulation

## Particle State

$$\mathbf{Y} = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

$$\mathbf{Y} = \begin{matrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{matrix}$$

## Particle Motion

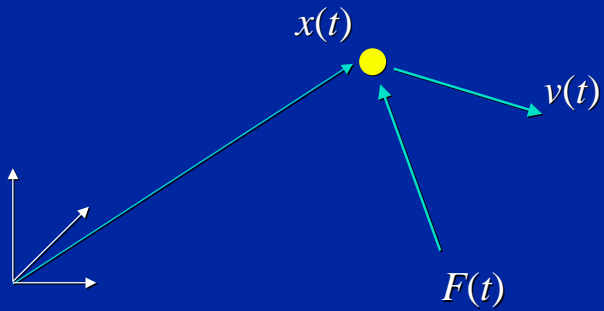


## State Derivative

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$

$$\frac{d}{dt} \mathbf{Y} = \begin{matrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{matrix}$$

## Particle Dynamics

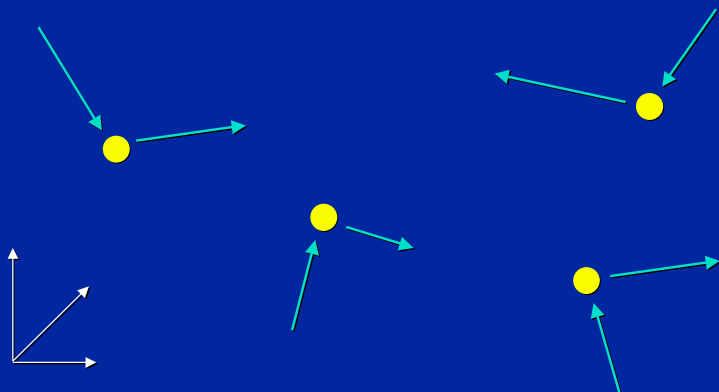


## State Derivative

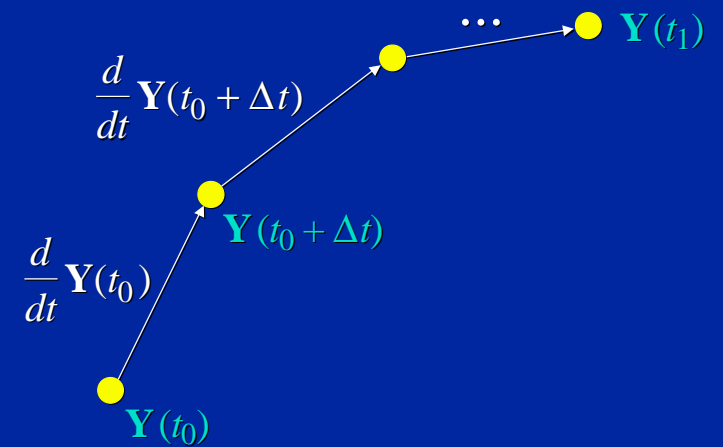
$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x_1(t) \\ v_1(t) \\ \times \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \times \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

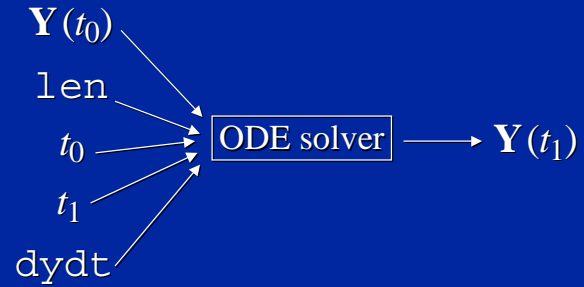
$$\frac{d}{dt} \mathbf{Y} = \begin{matrix} \boxed{\phantom{0}} & \boxed{\phantom{0}} & \dots & \text{6n elements} & \dots & \boxed{\phantom{0}} & \boxed{\phantom{0}} \end{matrix}$$

## Multiple Particles

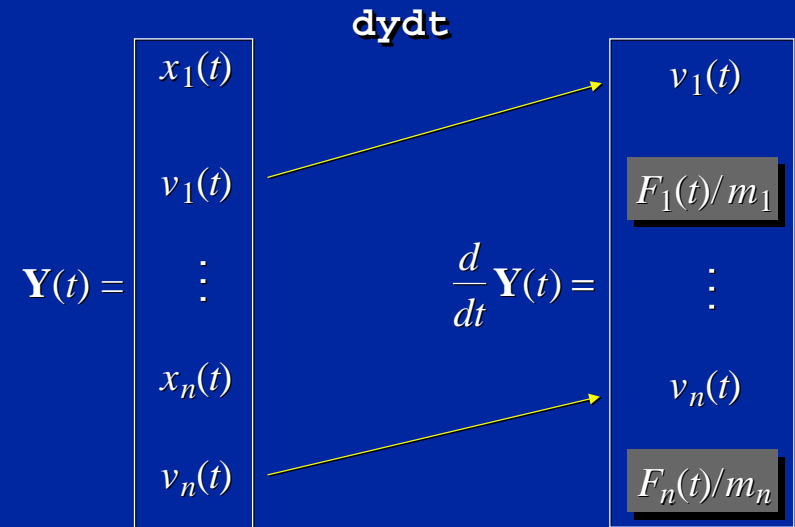


## ODE solution

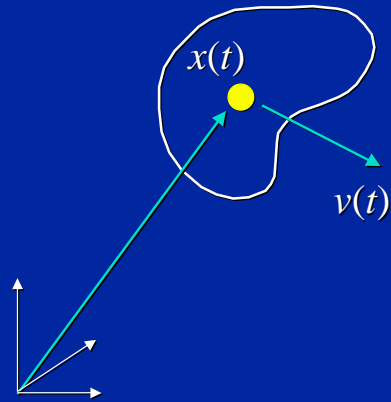




```
void dydt(double t, double y[],
          double ydot[])
```



### Rigid Body State

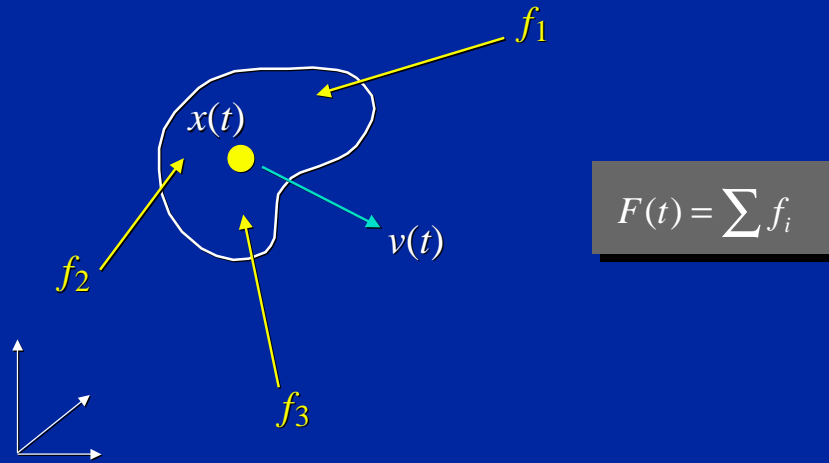


$$\mathbf{Y} = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

### Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ ? \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

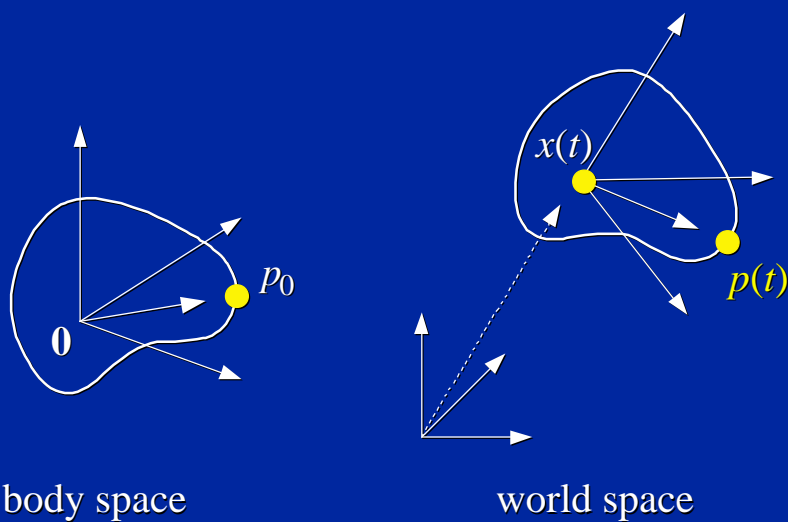
## Net Force



## Orientation

We represent orientation as a rotation matrix  $R(t)$ . Points are transformed from body-space to world-space as:

$$p(t) = R(t)p_0 + x(t)$$

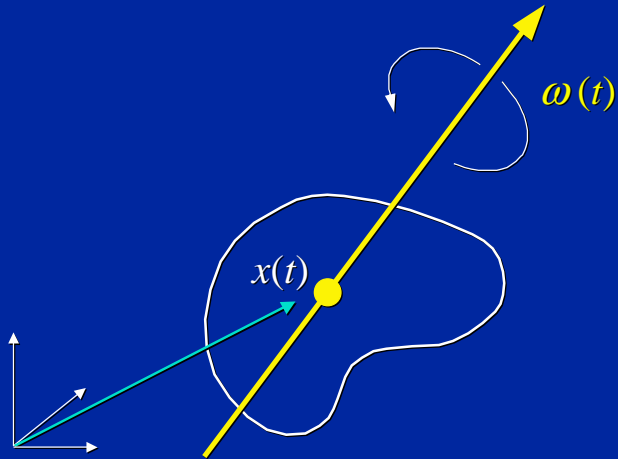


## Angular Velocity

We represent angular velocity as a vector  $\omega(t)$ , which encodes both the axis of the spin and the speed of the spin.

How are  $R(t)$  and  $\omega(t)$  related?

## Angular Velocity Definition



## Angular Velocity

$\dot{R}(t)$  and  $\omega(t)$  are related by

$$\frac{d}{dt} R(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} R(t)$$

$(\omega(t))^*$  is a shorthand for the above matrix)

## Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \langle \omega(t) \rangle \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate  $\dot{\omega}(t)$  and mass distribution to  $F(t)$ .

## Inertia Tensor

$$I(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

diagonal terms

off-diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2) dV$$

$$I_{xy} = -M \int_V xy dV$$

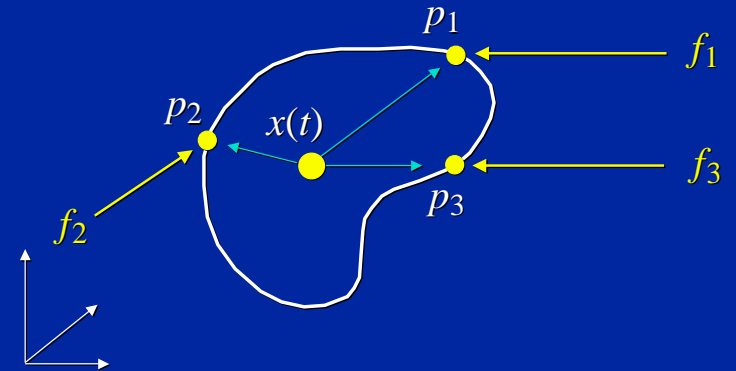
## Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ \boxed{Mv(t)} \\ \boxed{I(t)\omega(t)} \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$  – linear momentum

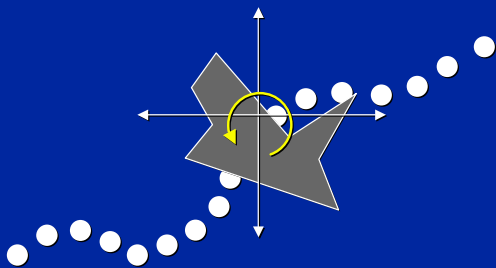
$L(t)$  – angular momentum

## Net Torque



$$\tau(t) = \sum (p_i - x(t)) \times f_i$$

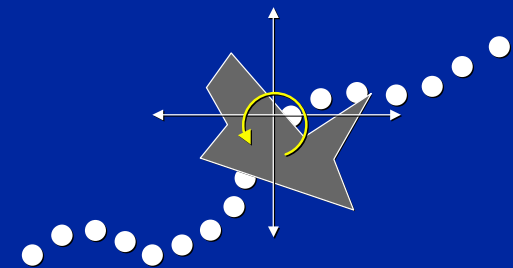
## Inertia Tensors Vary in World Space...



$$I_{xx} = M \int_V (y^2 + z^2) dV$$

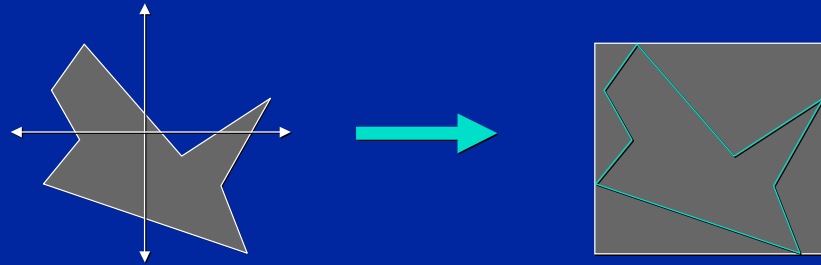
$$I_{xy} = -M \int_V xy dV$$

## ... but are Constant in Body Space



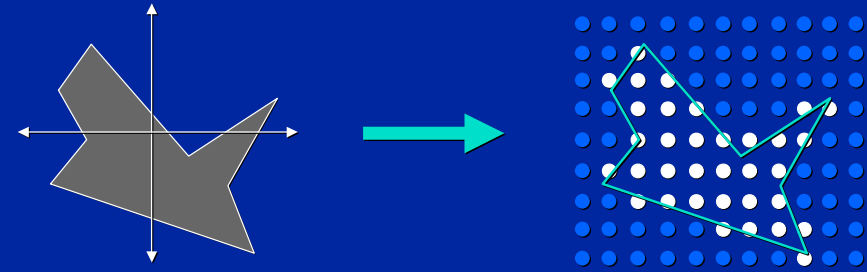
$$I(t) = R(t) I_{\text{body}} R(t)^T$$

### Approximating $I_{\text{body}}$ —Bounding Boxes



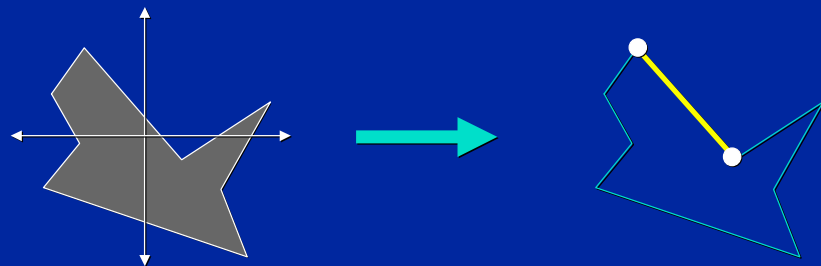
Pros: Simple.  
 Cons: Bounding box may not be a good fit.  
 Inaccurate.

### Approximating $I_{\text{body}}$ —Point Sampling



Pros: Simple, fairly accurate, no B-rep needed.  
 Cons: Expensive, requires volume test.

### Computing $I_{\text{body}}$ —Green's Theorem (Twice!)



Pros: Simple, exact, no volumes needed.  
 Cons: Requires B-rep.  
 Code: <http://www.acm.org/jgt/papers/Mirtich96>

### Rigid Body Equation of Motion

$$\frac{d}{dt} \mathbf{Y} = \frac{d}{dt} \begin{pmatrix} x(t) \\ R(t) \\ \boxed{Mv(t)} \\ \boxed{I(t)\omega(t)} \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$  – linear momentum

$L(t)$  – angular momentum