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CSE557 Final Project

Derivation Details for C2 continuous interpolating piecewise polynomial with local control.

```
> with( linalg ):
```

First, we demonstrate the computation of the weighting functions.

The following matrix will be evaluated for t1 = 0, t2 = 1 and then inverted to determine the unknown polynomial coefficients.

```
> A:=matrix( 6, 6, [ t1^5,t1^4,t1^3,t1^2,t1,1,5*t1^4,4*t1^3,3*t1^2,2*t1,1,0,20*t1^3,12*
,6*t1,2,0,0,t2^5,t2^4,t2^3,t2^2,t2,1,5*t2^4,4*t2^3,3*t2^2,2*t2,1,0,20*
t2^3,12*t2^2,6*t2,2,0,0] );
```

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 20 & 12 & 6 & 2 & 0 & 0 \end{bmatrix}$$

The following constraints are for lambda.

```
> b:=matrix( 6, 1, [1, 0, 0, 0, 0, 0] );
```

$$b := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> t1:=0;
```

$$t1 := 0$$

```
> t2:=1;
```

$$t2 := 1$$

We solve the system to determine the coefficients in lambda.

```
> lamCoeffs:=multiply( inverse( A ), b );
```

$$lamCoeffs := \begin{bmatrix} -6 \\ 15 \\ -10 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These are the constraints on mu.

```
> b:=matrix( 6, 1, [0, 0, 0, 1, 0, 0] );
```

$$b := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Solve.

```
> muCoeffs:=multiply( inverse( A ), b );
```

$$\mathit{muCoeffs} := \begin{bmatrix} 6 \\ -15 \\ 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we show the construction of the quadratic segments.

First we map the incoming time t to a normalized traversal distance w along the current line segment.

```
> #w:=(x3 - x) / (x3 - x2);
```

Next, we compute the uniformity of the left quadratic data.

```
#alpha:=(x2 - x1) / (x3 - x1)
```

We re-parameterize the left quadratic such that r goes from x2 to x3 as w goes from 0 to 1.

```
> r:= (1 - alpha ) * w + alpha;
```

$$r := \frac{(1-\alpha)\,(\mathit{x3}-x)}{\mathit{x3}-\mathit{x2}} + \alpha$$

The following computations setup the matrix computation of the general left quadratic.

```
> vec1:=matrix( 1, 3, [r^2, r, 1] ) ;
```

$$\mathit{vec1} := \left[\begin{array}{ccc} (\frac{(1-\alpha)\,(\mathit{x3}-x)}{\mathit{x3}-\mathit{x2}} + \alpha)^2 & \frac{(1-\alpha)\,(\mathit{x3}-x)}{\mathit{x3}-\mathit{x2}} + \alpha & 1 \end{array}\right]$$

```
> temp:=alpha;
```

$$\mathit{temp} := \alpha$$

```
> mat:=matrix( 3, 3, [1 / temp, -1 / (temp * (1 - temp)), 1 / (1
- temp), -(1 + temp) / temp, 1 / (temp * (1 - temp)), -temp / (1
- temp), 1, 0, 0]);
```

$$mat := \begin{bmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha\,(1-\alpha)} & \frac{1}{1-\alpha} \\ -\frac{1+\alpha}{\alpha} & \frac{1}{\alpha\,(1-\alpha)} & -\frac{\alpha}{1-\alpha} \\ 1 & 0 & 0 \end{bmatrix}$$

```
> vec2:=matrix( 3, 1, [y1, y2, y3] );
```

$$vec2 := \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

Finally, we compute p(r).

```
> p:=multiply( multiply( vec1, mat ), vec2 );
```

$$p := \left[\; (\frac{\%1^2}{\alpha} - \frac{\%1\,(1+\alpha)}{\alpha} + 1)\, y1 + (-\frac{\%1^2}{\alpha\,(1-\alpha)} + \frac{\%1}{\alpha\,(1-\alpha)})\, y2 + (\frac{\%1^2}{1-\alpha} - \frac{\%1\,\alpha}{1-\alpha})\, y3 \;\right]$$

$$\%1 := \frac{(1-\alpha)\,(x3-x)}{x3-x2} + \alpha$$

Again, we adjust for non-uniform spacing a re-parameterize the right quadratic to span the interval from x2 to x3.

```
> #beta:=(x3 - x2) / (x4 - x2);
> s:=w * beta;
```

$$s := \frac{(x3-x)\,\beta}{x3-x2}$$

```
> vec1:=matrix( 1, 3, [s^2, s, 1] ) ;
```

$$vec1 := \left[\; \frac{(x3-x)^2\,\beta^2}{(x3-x2)^2} \quad \frac{(x3-x)\,\beta}{x3-x2} \quad 1 \;\right]$$

```
> temp:=beta;
```

$$temp := \beta$$

```
> mat:=matrix( 3, 3, [1 / temp, -1 / (temp * (1 - temp)), 1 / (1
- temp), -(1 + temp) / temp, 1 / (temp * (1 - temp)), -temp / (1
- temp), 1, 0, 0]);
```

$$mat := \begin{bmatrix} \frac{1}{\beta} & -\frac{1}{\beta\,(1-\beta)} & \frac{1}{1-\beta} \\ -\frac{1+\beta}{\beta} & \frac{1}{\beta\,(1-\beta)} & -\frac{\beta}{1-\beta} \\ 1 & 0 & 0 \end{bmatrix}$$

```
> vec2:=matrix( 3, 1, [y2, y3, y4] );
```

$$vec2 := \begin{bmatrix} y2 \\ y3 \\ y4 \end{bmatrix}$$

And we solve for the right quadratic q(s).

```
> q:=multiply( multiply( vec1, mat ), vec2 );
```

$$q := \Big[\big(\frac{(x3-x)^2\,\beta}{(x3-x2)^2} - \frac{(x3-x)\,(1+\beta)}{x3-x2} + 1 \big)\,y2 + \big(-\frac{(x3-x)^2\,\beta}{(x3-x2)^2\,(1-\beta)} + \frac{x3-x}{(x3-x2)\,(1-\beta)} \big)\,y3 + \big(\frac{(x3-x)^2\,\beta^2}{(x3-x2)^2\,(1-\beta)} - \frac{(x3-x)\,\beta^2}{(x3-x2)\,(1-\beta)} \big)\,y4 \Big]$$

```
> vec1:=matrix( 1, 6, [w^5, w^4, w^3, w^2, w, 1] );
```

$$vec1 := \begin{bmatrix} \frac{(x3-x)^5}{(x3-x2)^5} & \frac{(x3-x)^4}{(x3-x2)^4} & \frac{(x3-x)^3}{(x3-x2)^3} & \frac{(x3-x)^2}{(x3-x2)^2} & \frac{x3-x}{x3-x2} & 1 \end{bmatrix}$$

```
> lambda:=multiply( vec1, lamCoeffs );
```

$$\lambda := \begin{bmatrix} -6\,\frac{(x3-x)^5}{(x3-x2)^5} + 15\,\frac{(x3-x)^4}{(x3-x2)^4} - 10\,\frac{(x3-x)^3}{(x3-x2)^3} + 1 \end{bmatrix}$$

```
> part1:=multiply( p, lambda );
```

$$part1 := \Big[\big(\big(\frac{\%1^2}{\alpha} - \frac{\%1\,(1+\alpha)}{\alpha} + 1 \big)\,y1 + \big(-\frac{\%1^2}{\alpha\,(1-\alpha)} + \frac{\%1}{\alpha\,(1-\alpha)} \big)\,y2 + \big(\frac{\%1^2}{1-\alpha} - \frac{\%1\,\alpha}{1-\alpha} \big)\,y3 \big) \big(-6\,\frac{(x3-x)^5}{(x3-x2)^5} + 15\,\frac{(x3-x)^4}{(x3-x2)^4} - 10\,\frac{(x3-x)^3}{(x3-x2)^3} + 1 \big) \Big]$$

$$\%1 := \frac{(1-\alpha)\,(x3-x)}{x3-x2} + \alpha$$

```
> part1:=part1[1,1];
```

$$part1 := \big(\big(\frac{\%1^2}{\alpha} - \frac{\%1\,(1+\alpha)}{\alpha} + 1 \big)\,y1 + \big(-\frac{\%1^2}{\alpha\,(1-\alpha)} + \frac{\%1}{\alpha\,(1-\alpha)} \big)\,y2 + \big(\frac{\%1^2}{1-\alpha} - \frac{\%1\,\alpha}{1-\alpha} \big)\,y3 \big) \big(-6\,\frac{(x3-x)^5}{(x3-x2)^5} + 15\,\frac{(x3-x)^4}{(x3-x2)^4} - 10\,\frac{(x3-x)^3}{(x3-x2)^3} + 1 \big)$$

$$\%1 := \frac{(1-\alpha)\,(x3-x)}{x3-x2} + \alpha$$

```
> mu:=multiply( vec1, muCoeffs );
```

$$\mu := \begin{bmatrix} 6\,\frac{(x3-x)^5}{(x3-x2)^5} - 15\,\frac{(x3-x)^4}{(x3-x2)^4} + 10\,\frac{(x3-x)^3}{(x3-x2)^3} \end{bmatrix}$$

```
> part2:=multiply( q, mu );
```

$$
\begin{aligned}
part2 :=\ & \Big[((\frac{(x3-x)^2\,\beta}{(x3-x2)^2} - \frac{(x3-x)\,(1+\beta)}{x3-x2} + 1)\, y2 \\
& + (-\frac{(x3-x)^2\,\beta}{(x3-x2)^2\,(1-\beta)} + \frac{x3-x}{(x3-x2)\,(1-\beta)})\, y3 \\
& + (\frac{(x3-x)^2\,\beta^2}{(x3-x2)^2\,(1-\beta)} - \frac{(x3-x)\,\beta^2}{(x3-x2)\,(1-\beta)})\, y4) \\
& (6\,\frac{(x3-x)^5}{(x3-x2)^5} - 15\,\frac{(x3-x)^4}{(x3-x2)^4} + 10\,\frac{(x3-x)^3}{(x3-x2)^3})\Big]
\end{aligned}
$$

```
> part2:=part2[1,1];
```

$$
\begin{aligned}
part2 :=\ & ((\frac{(x3-x)^2\,\beta}{(x3-x2)^2} - \frac{(x3-x)\,(1+\beta)}{x3-x2} + 1)\, y2 \\
& + (-\frac{(x3-x)^2\,\beta}{(x3-x2)^2\,(1-\beta)} + \frac{x3-x}{(x3-x2)\,(1-\beta)})\, y3 \\
& + (\frac{(x3-x)^2\,\beta^2}{(x3-x2)^2\,(1-\beta)} - \frac{(x3-x)\,\beta^2}{(x3-x2)\,(1-\beta)})\, y4) \\
& (6\,\frac{(x3-x)^5}{(x3-x2)^5} - 15\,\frac{(x3-x)^4}{(x3-x2)^4} + 10\,\frac{(x3-x)^3}{(x3-x2)^3})
\end{aligned}
$$

```
> final:=part1 + part2;
```

$$
\begin{aligned}
final :=\ & ((\frac{\%1^2}{\alpha} - \frac{\%1\,(1+\alpha)}{\alpha} + 1)\, y1 + (-\frac{\%1^2}{\alpha\,(1-\alpha)} + \frac{\%1}{\alpha\,(1-\alpha)})\, y2 + (\frac{\%1^2}{1-\alpha} - \frac{\%1\,\alpha}{1-\alpha})\, y3 \\
&)(-6\,\frac{(x3-x)^5}{(x3-x2)^5} + 15\,\frac{(x3-x)^4}{(x3-x2)^4} - 10\,\frac{(x3-x)^3}{(x3-x2)^3} + 1) + (\\
& (\frac{(x3-x)^2\,\beta}{(x3-x2)^2} - \frac{(x3-x)\,(1+\beta)}{x3-x2} + 1)\, y2 \\
& + (-\frac{(x3-x)^2\,\beta}{(x3-x2)^2\,(1-\beta)} + \frac{x3-x}{(x3-x2)\,(1-\beta)})\, y3 \\
& + (\frac{(x3-x)^2\,\beta^2}{(x3-x2)^2\,(1-\beta)} - \frac{(x3-x)\,\beta^2}{(x3-x2)\,(1-\beta)})\, y4) \\
& (6\,\frac{(x3-x)^5}{(x3-x2)^5} - 15\,\frac{(x3-x)^4}{(x3-x2)^4} + 10\,\frac{(x3-x)^3}{(x3-x2)^3}) \\
& \%1 := \frac{(1-\alpha)\,(x3-x)}{x3-x2} + \alpha
\end{aligned}
$$

We put this final expression in factored form. This is c(w), the local parametric curve defined in terms of the local parameter w.

```
> c:=collect( final, w );
```

We go ahead and claim the first and second derivatives via the chain rule.

```
> dcdx:=diff( c, x ) / (x3 - x2);
```

$$dcdx := \Bigg(-7\,\frac{(6\,\frac{(1-\alpha)\,y2}{\alpha}-6\,\frac{(1-\alpha)^2\,y1}{\alpha}-6\,(1-\alpha)\,y3+6\,\beta\,y2-6\,\frac{\beta\,y3}{1-\beta}+6\,\frac{\beta^2\,y4}{1-\beta})\,(x3-x)^6}{(x3-x2)^7}$$

$$-6((6\,(-1-\beta)\,y2+6\,\frac{y3}{1-\beta}-21\,\frac{\beta^2\,y4}{1-\beta}-15\,\beta\,y2+15\,\frac{\beta\,y3}{1-\beta}-6\,\%2-6\,\alpha\,y3$$

$$-6\,\%1-15\,\frac{(1-\alpha)\,y2}{\alpha}+15\,\frac{(1-\alpha)^2\,y1}{\alpha}+15\,(1-\alpha)\,y3)(x3-x)^5)\Big/(x3-x2)^6$$

$$-5((6\,y2-15\,(-1-\beta)\,y2-15\,\frac{y3}{1-\beta}+25\,\frac{\beta^2\,y4}{1-\beta}+10\,\beta\,y2-10\,\frac{\beta\,y3}{1-\beta}$$

$$-6\,(-\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha})\,y2+15\,\%2+15\,\alpha\,y3+15\,\%1+10\,\frac{(1-\alpha)\,y2}{\alpha}$$

$$-10\,\frac{(1-\alpha)^2\,y1}{\alpha}-10\,(1-\alpha)\,y3)(x3-x)^4)\Big/(x3-x2)^5-4((-15\,y2$$

$$+10\,(-1-\beta)\,y2+10\,\frac{y3}{1-\beta}-10\,\frac{\beta^2\,y4}{1-\beta}+15\,(-\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha})\,y2-10\,\%2$$

$$-10\,\alpha\,y3-10\,\%1)(x3-x)^3)\Big/(x3-x2)^4$$

$$-3\,\frac{(-10\,(-\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha})\,y2+10\,y2)\,(x3-x)^2}{(x3-x2)^3}$$

$$-2\,\frac{(-\frac{(1-\alpha)\,y2}{\alpha}+\frac{(1-\alpha)^2\,y1}{\alpha}+(1-\alpha)\,y3)\,(x3-x)}{(x3-x2)^2}-\frac{\%2+\alpha\,y3+\%1}{x3-x2}\Bigg)\,/($$

$$x3-x2)$$

$$\%1 := (-\frac{2-2\,\alpha}{1-\alpha}+\frac{1}{\alpha})\,y2$$

$$\%2 := (-\frac{(1-\alpha)\,(1+\alpha)}{\alpha}+2-2\,\alpha)\,y1$$

```
> d2cdx2:=diff( c, x ) / (x3 - x2);
```

$$d2cdx2 := \Bigg(-7\,\frac{(6\,\frac{(1-\alpha)\,y2}{\alpha}-6\,\frac{(1-\alpha)^2\,y1}{\alpha}-6\,(1-\alpha)\,y3+6\,\beta\,y2-6\,\frac{\beta\,y3}{1-\beta}+6\,\frac{\beta^2\,y4}{1-\beta})\,(x3-x)^6}{(x3-x2)^7}$$

$$-6((6\,(-1-\beta)\,y2+6\,\frac{y3}{1-\beta}-21\,\frac{\beta^2\,y4}{1-\beta}-15\,\beta\,y2+15\,\frac{\beta\,y3}{1-\beta}-6\,\%2-6\,\alpha\,y3$$

$$-6\,\%1-15\,\frac{(1-\alpha)\,y2}{\alpha}+15\,\frac{(1-\alpha)^2\,y1}{\alpha}+15\,(1-\alpha)\,y3)(x3-x)^5)\Big/(x3-x2)^6$$

$$-5((6\,y2-15\,(-1-\beta)\,y2-15\,\frac{y3}{1-\beta}+25\,\frac{\beta^2\,y4}{1-\beta}+10\,\beta\,y2-10\,\frac{\beta\,y3}{1-\beta}$$

$$-6\,(-\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha})\,y2+15\,\%2+15\,\alpha\,y3+15\,\%1+10\,\frac{(1-\alpha)\,y2}{\alpha}$$

$$- 10\,\frac{(1-\alpha)^2\,y1}{\alpha} - 10\,(1-\alpha)\,y3)(x3-x)^4)\Big/(x3-x2)^5 - 4((-15\,y2$$

$$+ 10\,(-1-\beta)\,y2 + 10\,\frac{y3}{1-\beta} - 10\,\frac{\beta^2\,y4}{1-\beta} + 15\,(-\frac{\alpha}{1-\alpha} + \frac{1}{1-\alpha})\,y2 - 10\,\%2$$

$$- 10\,\alpha\,y3 - 10\,\%1)(x3-x)^3)\Big/(x3-x2)^4$$

$$- 3\,\frac{(-10\,(-\frac{\alpha}{1-\alpha} + \frac{1}{1-\alpha})\,y2 + 10\,y2)\,(x3-x)^2}{(x3-x2)^3}$$

$$- 2\,\frac{(-\frac{(1-\alpha)\,y2}{\alpha} + \frac{(1-\alpha)^2\,y1}{\alpha} + (1-\alpha)\,y3)\,(x3-x)}{(x3-x2)^2} - \frac{\%2 + \alpha\,y3 + \%1}{x3-x2}\Bigg)\Big/($$

$$x3-x2)$$

$$\%1 := (-\frac{2-2\,\alpha}{1-\alpha} + \frac{1}{\alpha})\,y2$$

$$\%2 := (-\frac{(1-\alpha)\,(1+\alpha)}{\alpha} + 2 - 2\,\alpha)\,y1$$

We now bolster these claims with an explicit change of variables.
The first derivative is equivalent to:

```
>  subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1) / (x3 - x1),
beta = (x3 - x2) / (x4 - x2)}, diff( c, x ) );
```

$$-7\,\Bigg(\Bigg(6\,\%4 - 6\,\%3 - 6\,(1 - \frac{x2-x1}{x3-x1})\,y3 + 6\,\frac{(x3-x2)\,y2}{x4-x2} - 6\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})}$$

$$+ 6\,\%6\Bigg)\,(x3-x)^6\Bigg)\Big/(x3-x2)^7 - 6\,\Bigg(\Bigg(6\,(-1-\frac{x3-x2}{x4-x2})\,y2 + 6\,\frac{y3}{1-\frac{x3-x2}{x4-x2}} - 21\,\%6$$

$$- 15\,\frac{(x3-x2)\,y2}{x4-x2} + 15\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})} - 6\,\%2 - 6\,\frac{(x2-x1)\,y3}{x3-x1} - 6\,\%1$$

$$- 15\,\%4 + 15\,\%3 + 15\,(1-\frac{x2-x1}{x3-x1})\,y3\Bigg)\,(x3-x)^5\Bigg)\Big/(x3-x2)^6 - 5\,\Bigg(\Bigg(6\,y2$$

$$- 15\,(-1-\frac{x3-x2}{x4-x2})\,y2 - 15\,\frac{y3}{1-\frac{x3-x2}{x4-x2}} + 25\,\%6 + 10\,\frac{(x3-x2)\,y2}{x4-x2}$$

$$- 10\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})} - 6\,\%5 + 15\,\%2 + 15\,\frac{(x2-x1)\,y3}{x3-x1} + 15\,\%1$$

$$+ 10\,\%4 - 10\,\%3 - 10\,(1-\frac{x2-x1}{x3-x1})\,y3\Bigg)\,(x3-x)^4\Bigg)\Big/(x3-x2)^5 - 4\,\Bigg(\Bigg(-15\,y2$$

$$+ 10 \left(-1 - \frac{x3 - x2}{x4 - x2}\right) y2 + 10 \frac{y3}{1 - \frac{x3 - x2}{x4 - x2}} - 10 \%6 + 15 \%5 - 10 \%2$$

$$\left. \left. - 10 \frac{(x2 - x1)\, y3}{x3 - x1} - 10 \%1 \right) (x3 - x)^3 \right) \Big/ (x3 - x2)^4 - 3 \frac{(-10 \%5 + 10\, y2)\, (x3 - x)^2}{(x3 - x2)^3}$$

$$- 2 \frac{(-\%4 + \%3 + (1 - \frac{x2 - x1}{x3 - x1})\, y3)\, (x3 - x)}{(x3 - x2)^2} - \frac{\%2 + \frac{(x2 - x1)\, y3}{x3 - x1} + \%1}{x3 - x2}$$

$$\%1 := \left(- \frac{2 - 2 \frac{x2 - x1}{x3 - x1}}{1 - \frac{x2 - x1}{x3 - x1}} + \frac{x3 - x1}{x2 - x1} \right) y2$$

$$\%2 := \left(- \frac{(1 - \frac{x2 - x1}{x3 - x1})\, (1 + \frac{x2 - x1}{x3 - x1})\, (x3 - x1)}{x2 - x1} + 2 - 2 \frac{x2 - x1}{x3 - x1} \right) y1$$

$$\%3 := \frac{(1 - \frac{x2 - x1}{x3 - x1})^2\, (x3 - x1)\, y1}{x2 - x1}$$

$$\%4 := \frac{(1 - \frac{x2 - x1}{x3 - x1})\, (x3 - x1)\, y2}{x2 - x1}$$

$$\%5 := \left(- \frac{x2 - x1}{(x3 - x1)\, (1 - \frac{x2 - x1}{x3 - x1})} + \frac{1}{1 - \frac{x2 - x1}{x3 - x1}} \right) y2$$

$$\%6 := \frac{(x3 - x2)^2\, y4}{(x4 - x2)^2\, (1 - \frac{x3 - x2}{x4 - x2})}$$

And the second derivative is:

```
>  subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1) / (x3 - x1),
beta = (x3 - x2) / (x4 - x2)}, diff( c, x$2 ) );
```

$$42 \left(\left(6 \%2 - 6 \%1 - 6 \left(1 - \frac{x2 - x1}{x3 - x1}\right) y3 + 6 \frac{(x3 - x2)\, y2}{x4 - x2} - 6 \frac{(x3 - x2)\, y3}{(x4 - x2)\, (1 - \frac{x3 - x2}{x4 - x2})} \right. \right.$$

$$\left. \left. + 6 \%5 \right) (x3 - x)^5 \right) \Big/ (x3 - x2)^7 + 30 \left(\left(6 \left(-1 - \frac{x3 - x2}{x4 - x2}\right) y2 + 6 \frac{y3}{1 - \frac{x3 - x2}{x4 - x2}} - 21 \%5 \right. \right.$$

$$- 15 \frac{(x3 - x2)\, y2}{x4 - x2} + 15 \frac{(x3 - x2)\, y3}{(x4 - x2)\, (1 - \frac{x3 - x2}{x4 - x2})} - 6 \%4 - 6 \frac{(x2 - x1)\, y3}{x3 - x1}$$

$$-6\left(-\frac{2-2\,\frac{x2-x1}{x3-x1}}{1-\frac{x2-x1}{x3-x1}}+\frac{x3-x1}{x2-x1}\right)\,y2-15\,\%2+15\,\%1+15\,(1-\frac{x2-x1}{x3-x1})\,y3\Bigg)$$

$$(x3-x)^4\Bigg)\Big/(x3-x2)^6+20\,\Bigg(\Bigg(6\,y2-15\,(-1-\frac{x3-x2}{x4-x2})\,y2-15\,\frac{y3}{1-\frac{x3-x2}{x4-x2}}$$

$$+25\,\%5+10\,\frac{(x3-x2)\,y2}{x4-x2}-10\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})}-6\,\%3+15\,\%4$$

$$+15\,\frac{(x2-x1)\,y3}{x3-x1}+15\,\left(-\frac{2-2\,\frac{x2-x1}{x3-x1}}{1-\frac{x2-x1}{x3-x1}}+\frac{x3-x1}{x2-x1}\right)\,y2+10\,\%2-10\,\%1$$

$$-10\,(1-\frac{x2-x1}{x3-x1})\,y3\Bigg)\,(x3-x)^3\Bigg)\Big/(x3-x2)^5+12\,\Bigg(\Bigg(-15\,y2+10\,(-1-\frac{x3-x2}{x4-x2})\,y2$$

$$+10\,\frac{y3}{1-\frac{x3-x2}{x4-x2}}-10\,\%5+15\,\%3-10\,\%4-10\,\frac{(x2-x1)\,y3}{x3-x1}$$

$$-10\,\left(-\frac{2-2\,\frac{x2-x1}{x3-x1}}{1-\frac{x2-x1}{x3-x1}}+\frac{x3-x1}{x2-x1}\right)\,y2\Bigg)\,(x3-x)^2\Bigg)\Big/(x3-x2)^4$$

$$+6\,\frac{(-10\,\%3+10\,y2)\,(x3-x)}{(x3-x2)^3}+2\,\frac{-\%2+\%1+(1-\frac{x2-x1}{x3-x1})\,y3}{(x3-x2)^2}$$

$$\%1:=\frac{(1-\frac{x2-x1}{x3-x1})^2\,(x3-x1)\,y1}{x2-x1}$$

$$\%2:=\frac{(1-\frac{x2-x1}{x3-x1})\,(x3-x1)\,y2}{x2-x1}$$

$$\%3:=\left(-\frac{x2-x1}{(x3-x1)\,(1-\frac{x2-x1}{x3-x1})}+\frac{1}{1-\frac{x2-x1}{x3-x1}}\right)\,y2$$

$$\%4:=\left(-\frac{(1-\frac{x2-x1}{x3-x1})\,(1+\frac{x2-x1}{x3-x1})\,(x3-x1)}{x2-x1}+2-2\,\frac{x2-x1}{x3-x1}\right)\,y1$$

$$\%5:=\frac{(x3-x2)^2\,y4}{(x4-x2)^2\,(1-\frac{x3-x2}{x4-x2})}$$

```
> subs( {x = x2, alpha = (x2 - x1) / (x3 - x1), beta = (x3 - x2)
/ (x4 - x2)}, % );
```

$$-7\,\frac{6\,\%4-6\,\%3-6\,(1-\frac{x2-x1}{x3-x1})\,y3+6\,\frac{(x3-x2)\,y2}{x4-x2}-6\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})}+6\,\%6}{x3-x2}$$

$$-6\left(6\,(-1-\frac{x3-x2}{x4-x2})\,y2+6\,\frac{y3}{1-\frac{x3-x2}{x4-x2}}-21\,\%6-15\,\frac{(x3-x2)\,y2}{x4-x2}\right.$$

$$+15\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})}-6\,\%2-6\,\frac{(x2-x1)\,y3}{x3-x1}-6\,\%1-15\,\%4+15\,\%3$$

$$\left.+15\,(1-\frac{x2-x1}{x3-x1})\,y3\right)/(x3-x2)-5\left(6\,y2-15\,(-1-\frac{x3-x2}{x4-x2})\,y2-15\,\frac{y3}{1-\frac{x3-x2}{x4-x2}}\right.$$

$$+25\,\%6+10\,\frac{(x3-x2)\,y2}{x4-x2}-10\,\frac{(x3-x2)\,y3}{(x4-x2)\,(1-\frac{x3-x2}{x4-x2})}-6\,\%5+15\,\%2$$

$$\left.+15\,\frac{(x2-x1)\,y3}{x3-x1}+15\,\%1+10\,\%4-10\,\%3-10\,(1-\frac{x2-x1}{x3-x1})\,y3\right)/(x3-x2)-4\,\Bigg($$

$$-15\,y2+10\,(-1-\frac{x3-x2}{x4-x2})\,y2+10\,\frac{y3}{1-\frac{x3-x2}{x4-x2}}-10\,\%6+15\,\%5-10\,\%2$$

$$\left.-10\,\frac{(x2-x1)\,y3}{x3-x1}-10\,\%1\right)/(x3-x2)-3\,\frac{-10\,\%5+10\,y2}{x3-x2}$$

$$-2\,\frac{-\%4+\%3+(1-\frac{x2-x1}{x3-x1})\,y3}{x3-x2}-\frac{\%2+\frac{(x2-x1)\,y3}{x3-x1}+\%1}{x3-x2}$$

$$\%1:=\left(-\frac{2-2\,\frac{x2-x1}{x3-x1}}{1-\frac{x2-x1}{x3-x1}}+\frac{x3-x1}{x2-x1}\right)\,y2$$

$$\%2:=\left(-\frac{(1-\frac{x2-x1}{x3-x1})\,(1+\frac{x2-x1}{x3-x1})\,(x3-x1)}{x2-x1}+2-2\,\frac{x2-x1}{x3-x1}\right)\,y1$$

$$\%3:=\frac{(1-\frac{x2-x1}{x3-x1})^2\,(x3-x1)\,y1}{x2-x1}$$

$$\%4:=\frac{(1-\frac{x2-x1}{x3-x1})\,(x3-x1)\,y2}{x2-x1}$$

$$\%5 := \left(-\frac{x2 - x1}{(x3 - x1)\,(1 - \frac{x2 - x1}{x3 - x1})} + \frac{1}{1 - \frac{x2 - x1}{x3 - x1}}\right) y2$$

$$\%6 := \frac{(x3 - x2)^2\, y4}{(x4 - x2)^2\,(1 - \frac{x3 - x2}{x4 - x2})}$$

We now consider the claim of Cˆ2 continuity.

We do this by evaluating the derivative at the right end point.

```
>  simplify( subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1)
/ (x3 - x1), beta = (x3 - x2) / (x4 - x2), x = x3}, diff( c, x )
) );
```

$$-(y1\,x3^2 + y1\,x2^2 - 2\,y1\,x3\,x2 - y3\,x2^2 + 2\,y3\,x2\,x1 - y3\,x1^2 + 2\,y2\,x3\,x2 - y2\,x3^2 - 2\,y2\,x2\,x1 + y2\,x1^2)/((-x3 + x2)\,(x3 - x1)\,(x2 - x1))$$

And the left end point.

```
>  simplify( subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1)
/ (x3 - x1), beta = (x3 - x2) / (x4 - x2), x = x2}, diff( c, x )
) );
```

$$-(-y4\,x2^2 + y3\,x2^2 - 2\,y3\,x3\,x2 + 2\,x3\,y4\,x2 + y2\,x3^2 - y4\,x3^2 + y2\,x4^2 - y3\,x4^2 + 2\,y3\,x4\,x3 - 2\,y2\,x4\,x3)/((-x3 + x2)\,(x4 - x2)\,(x4 - x3))$$

We see from this that the derivative at the right end point of this curve equals the derivative at the left end of the curve defined by the NEXT four points.

Similarly for the second derivative we observe

```
>  simplify( subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1)
/ (x3 - x1), beta = (x3 - x2) / (x4 - x2), x = x3}, diff( c, x$2
) ) );
```

$$2\,\frac{y2\,x3 - x1\,y2 - y1\,x3 + y1\,x2 - y3\,x2 + y3\,x1}{(-x3 + x2)\,(x3 - x1)\,(x2 - x1)}$$

```
>  simplify( subs( {w = (x3 - x) / (x3 - x2), alpha = (x2 - x1)
/ (x3 - x1), beta = (x3 - x2) / (x4 - x2), x = x2}, diff( c, x$2
) ) );
```

$$-2\,\frac{-y4\,x2 + y3\,x2 + y4\,x3 - y2\,x3 + y2\,x4 - y3\,x4}{(-x3 + x2)\,(x4 - x2)\,(x4 - x3)}$$

We see this agreement in the same way as with the first derivative. Hence the curve is Cˆ2.