Linear (and contextual) Bandits: Rich decision sets (and side information)

Sham M. Kakade

Machine Learning for Big Data
CSE547/STAT548

University of Washington

Announcements...

- Poster session: June 1, 9-11:30a
  - Request: CSE grad students, could you please help others with poster printing?
  - Aravind: Ask by 2p on Weds for help printing.
  - Prepare, at most, a 2 minute verbal summary.
  - Come earlier to setup.
  - Submit your poster on Canvas.
- Due Dates: Please be on time.

Today:

- review: Linear bandits
- today: contextual bandits, game trees?

Review

Bandits in practice: two major issues

- The decision space is very large.
  - Drug cocktails
  - Ad design
- We often have “side information” when making a decision
  - history of a user

More real motivations...

Clinical trials:





$\mathcal{B}(\mu_1)$ $\mathcal{B}(\mu_2)$ $\mathcal{B}(\mu_3)$ $\mathcal{B}(\mu_4)$ $\mathcal{B}(\mu_5)$

- choose a treatment A_t for patient t
- observe a response $X_t \in \{0, 1\} : \mathbb{P}(X_t = 1) = \mu_{A_t}$
- Goal: maximize the number of patient healed

Recommendation tasks:

ν_1 ν_2 ν_3 ν_4 ν_5

- recommend a movie A_t for visitor t
- observe a rating $X_t \sim \nu_{A_t}$ (e.g. $X_t \in \{1, \ldots, 5\}$)

Linear bandits

- An additive effects model.
- Suppose each round we take a decision $x \in \mathcal{D} \subset \mathcal{R}^d$.
  - x is paths on a graph.
  - x is a feature vector of properties of an ad
  - x is a which drugs are being taken
- Upon taking action x, we get reward r, with expectation:

$$\mathbb{E}[r|x] = \mu^\top x$$

- only d unknown parameters (and “effectively” 2^d actions)
- W desire an algorithm $\mathcal{A}$ (mapping histories to decisions), which has low regret.

$$T\mu^\top x_* - \sum_{t=1}^{T} \mathbb{E}[\mu^\top x_t|\mathcal{A}] \leq ??$$

(where x_* is the best decision)

Example: Shortest paths...

Algorithm Idea

- again, let's think of optimism in the face of uncertainty
- we observed some $r_1, \ldots r_{t-1}$, and have taken $x_1, \ldots x_{t-1}$.
- Questions:
  - what is an estimate of the reward of $\mathbb{E}[r|x]$ and what is our uncertainty?
  - what is an estimate of μ and what is our uncertainty?

Regression!

- Define:

$$A_t := \sum_{\tau<t} x_\tau x_\tau^\top + \lambda I, \;\; b_t := \sum_{\tau<t} x_\tau r_\tau$$

- Our estimate of μ

$$\hat{\mu}_t = A_t^{-1} b_t$$

- Confidence of our estimate:

$$\|\mu - \hat{\mu}_t\|_{A_t}^2 \le \mathcal{O}(d \log t)$$

LinUCB

- Again, optimism in the face of uncertainty.
- Define:

$$B_t := \{\nu | \|\nu - \hat{\mu}_t\|_{A_t}^2 \leq \mathcal{O} d \log t\}$$

- (Lin UCB) take action:

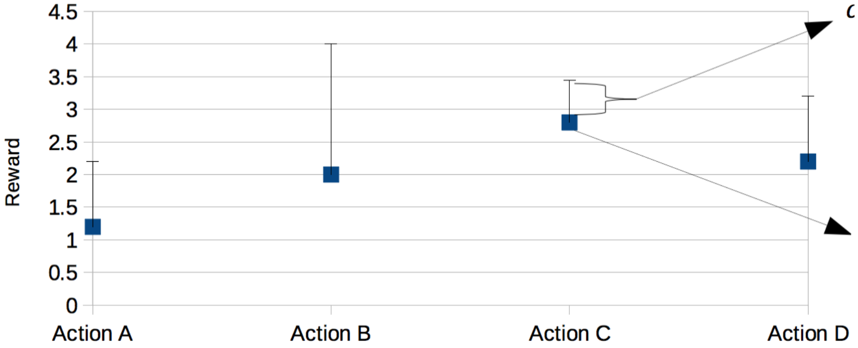
$$x_t = \mathrm{argmax}_{x \in \mathcal{D}} \max_{\nu \in B_t} \nu^\top x$$

  then update A_t, B_t, b_t, and $\hat{\mu}_t$.
- Equivalently, take action:

$$x_t = \mathrm{argmax}_{x \in \mathcal{D}} \ \hat{\mu}_t^\top x + (d \log t) \sqrt{x A_t^{-1} x}$$

LinUCB: Geometry

LinUCB: Confidence intervals



Today

LinUCB

- Regret bound of LinUCB

$$T\mu^\top x_* - \sum_{t=1}^{T} \mathbb{E}[\mu^\top x_t] \le {}^*(d\sqrt{T})$$

(this is the best possible, up to log factors).

- Compare to $O(\sqrt{KT})$
  - Independent of number of actions.
  - k-arm case is a special case.
- Thompson sampling: This is a good algorithm in practice.

Proof Idea...

- Stats: need to show that B_t is a valid confidence region.
- Geometric lemma: The regret is upper bounded by the:

$$\log \frac{\text{volume of posterior cov}}{\text{volume of prior cov}}$$

- Then just bound the worst case log volume change.

What about context?

Clinical trials:





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The Contextual Bandit Game

- Game: for $t = 1, 2, \ldots$
  - At each time t, we obtain context (e.g. side information, user information) c_t
  - Our feasible action set is A_t.
  - We choose arm $a_t \in A_t$ and receive reward r_{t,a_t}. (what assumptions on the reward process?)
- Goal: Algorithm $\mathcal{A}$ to have low regret:

$$\mathbb{E}[\sum_t (r_{t,a_t^*} - r_t)|\mathcal{A}] \leq ??$$

where $\mathbb{E}[r_{t,a_t^*}]$ is the optimal expected reward at time t.

How should we model outcomes?

- Example: ad (or movie, song, etc) prediction.
  What is prob. that a user u clicks on an ad a.
- How should we model the click probability of a for user u?
- Featurizations: suppose we have $\phi_{\text{ad}}(a) \in \mathcal{R}^{d_{\text{ad}}}$ and $\phi_{\text{user}}(u) \in \mathcal{R}^{d_{\text{user}}}$.
- We could make an "outer product" feature vector x as:

$$x(a,u) = \text{Vector}(\phi_{\text{ad}}(a)\phi_{\text{user}}(u)^\top) \in \mathcal{R}^{d_{\text{ad}} d_{\text{user}}}$$

- We could model the probabilities as:

$$\mathbb{E}[click = 1 | a, u] = \mu^\top x(a,u)$$

  (or log linear)
- How do we estimate μ?

Contextual Linear bandits

- Suppose each round t, we take a decision $x \in \mathcal{D}_t \subset \mathcal{R}^d$ (D_t may be time varying).
  - map each ad/user a to $x(a, u)$.
  - $D_t = \{x(a, u_t) | a \text{ is a feasible ad at time } t\}$
  - Our decision is a feature vector in $x \in D_t$.
- Upon taking action $x_t \in D_t$, we get reward r_t, with expectation:

$$\mathbb{E}[r_t | x_t \in D_t] = \mu^\top x_t$$

  (here μ is assumed constant over time).
- Our regret:

$$\mathbb{E}[\sum_t (\mu^\top x_{t,a_t^*} - \mu^\top x_t) | \mathcal{A}] \leq ??$$

  (where x_{t,a_t^*} is the best decision at time t)

Algorithm

- let's just run linUCB (or Thompson sampling)
- Nothing really changes:
  - A_t and b_t are the same updating rules
  - now our decision is:

$$x_t = \mathrm{argmax}_{x \in \mathcal{D}_t} \max_{\nu \in B_t} \nu^\top x$$

    i.e.

$$x_t = \mathrm{argmax}_{x \in \mathcal{D}_t} \;\; \hat{\mu}_t^\top x + (d \log t)\sqrt{x A_t^{-1} x}$$

- Regret bound is still $O(d\sqrt{T})$.

Acknowledgements

- http://gdrro.lip6.fr/sites/default/files/JourneeCOSdec2015-Kaufman.pdf
- https://sites.google.com/site/banditstutorial/
- http://www.yisongyue.com/courses/cs159/lectures/LinUCB.pdf