Linear (and contextual) Bandits:
Rich decision sets (and side information)

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Announcements...

- Poster session: June 1, 9-11:30a
  - **Request:** CSE grad students, could you please help others with poster printing?
  - Aravind: Ask by 2p on Weds for help printing.
  - Prepare, **at most**, a 2 minute verbal summary.
  - Come earlier to setup.
  - Submit your poster on Canvas.

- **Due Dates:** Please be on time.

**Today:**
- review: Linear bandits
- today: contextual bandits, game trees?
Review
Bandits in practice: two major issues

- The decision space is very large.
  - Drug cocktails
  - Ad design
- We often have “side information” when making a decision
  - history of a user
More real motivations...

Clinical trials:

\[ B(\mu_1) \quad B(\mu_2) \quad B(\mu_3) \quad B(\mu_4) \quad B(\mu_5) \]

- choose a treatment \( A_t \) for patient \( t \)
- observe a response \( X_t \in \{0, 1\} \): \( P(X_t = 1) = \mu_{A_t} \)
- **Goal**: maximize the number of patient healed

Recommendation tasks:

\[ \nu_1 \quad \nu_2 \quad \nu_3 \quad \nu_4 \quad \nu_5 \]

- recommend a movie \( A_t \) for visitor \( t \)
- observe a rating \( X_t \sim \nu_{A_t} \) (e.g. \( X_t \in \{1, \ldots, 5\} \))
Linear bandits

- An additive effects model.
- Suppose each round we take a decision \( x \in D \subset \mathcal{R}^d \).
  - \( x \) is paths on a graph.
  - \( x \) is a feature vector of properties of an ad
  - \( x \) is a which drugs are being taken
- Upon taking action \( x \), we get reward \( r \), with expectation:
  \[
  \mathbb{E}[r|x] = \mu^\top x
  \]
- Only \( d \) unknown parameters (and “effectively” \( 2^d \) actions)
- We desire an algorithm \( A \) (mapping histories to decisions), which has low regret.
  \[
  T \mu^\top x_* - \sum_{t=1}^{T} \mathbb{E}[\mu^\top x_t|A] \leq ??
  \]
  (where \( x_* \) is the best decision)
Example: Shortest paths...
again, let’s think of optimism in the face of uncertainty
we observed some \( r_1, \ldots r_{t-1} \), and have taken \( x_1, \ldots x_{t-1} \).

Questions:
- what is an estimate of the reward of \( \mathbb{E}[r|x] \) and what is our uncertainty?
- what is an estimate of \( \mu \) and what is our uncertainty?
Regression!

Define:

\[ A_t := \sum_{\tau < t} x_\tau x_\tau^\top + \lambda I, \quad b_t := \sum_{\tau < t} x_\tau r_\tau \]

Our estimate of \( \mu \)

\[ \hat{\mu}_t = A_t^{-1} b_t \]

Confidence of our estimate:

\[ \| \mu - \hat{\mu}_t \|^2_{A_t} \leq O(d \log t) \]
Again, optimism in the face of uncertainty.

Define:

\[ B_t := \{ \nu \parallel \nu - \hat{\mu}_t \parallel^2_{A_t} \leq O d \log t \} \]

(*Lin UCB*) take action:

\[ x_t = \arg\max_{x \in \mathcal{D}} \max_{\nu \in B_t} \nu^\top x \]

then update \( A_t, B_t, b_t, \) and \( \hat{\mu}_t. \)

Equivalently, take action:

\[ x_t = \arg\max_{x \in \mathcal{D}} \hat{\mu}_t^\top x + (d \log t) \sqrt{xA_t^{-1}x} \]
LinUCB: Geometry
LinUCB: Confidence intervals
Today
LinUCB

- Regret bound of LinUCB

\[ T \mu^\top x_\star - \sum_{t=1}^{T} \mathbb{E}[\mu^\top x_t] \leq ^* (d \sqrt{T}) \]

(this is the best possible, up to log factors).

- Compare to \( O(\sqrt{KT}) \)
  - Independent of number of actions.
  - \( k \)-arm case is a special case.

**Thompson sampling:** This is a good algorithm in practice.
Proof Idea...

- Stats: need to show that $B_t$ is a valid confidence region.
- Geometric lemma: The regret is upper bounded by the:

$$\log \frac{\text{volume of posterior cov}}{\text{volume of prior cov}}$$

- Then just bound the worst case log volume change.
What about context?

Clinical trials:

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The Contextual Bandit Game

- **Game:** for \( t = 1, 2, \ldots \)
  - At each time \( t \), we obtain context (e.g. side information, user information) \( c_t \)
  - Our feasible action set is \( A_t \).
  - We choose arm \( a_t \in A_t \) and receive reward \( r_{t,a_t} \).
    (what assumptions on the reward process?)

- **Goal:** Algorithm \( \mathcal{A} \) to have low regret:
  $$E\left[ \sum_t (r_{t,a_t^*} - r_t) | \mathcal{A} \right] \leq ??$$

  where \( E[r_{t,a_t^*}] \) is the optimal expected reward at time \( t \).
How should we model outcomes?

Example: ad (or movie, song, etc) prediction. What is prob. that a user $u$ clicks on an ad $a$.

How should we model the click probability of $a$ for user $u$?

Featurizations: suppose we have $\phi_{\text{ad}}(a) \in \mathcal{R}^{d_{\text{ad}}}$ and $\phi_{\text{user}}(u) \in \mathcal{R}^{d_{\text{user}}}$.

We could make an “outer product” feature vector $x$ as:

$$x(a, u) = \text{Vector}(\phi_{\text{ad}}(a)\phi_{\text{user}}(u)^\top) \in \mathcal{R}^{d_{\text{ad}}d_{\text{user}}}$$

We could model the probabilities as:

$$\mathbb{E}[\text{click} = 1|a, u] = \mu^\top x(a, u)$$

(or log linear)

How do we estimate $\mu$?
Suppose each round $t$, we take a decision $x \in D_t \subset \mathcal{R}^d$ ($D_t$ may be time varying).
- map each ad/user $a$ to $x(a, u)$.
- $D_t = \{x(a, u_t)|a \text{ is a feasible ad at time } t\}$
- Our decision is a feature vector in $x \in D_t$.

Upon taking action $x_t \in D_t$, we get reward $r_t$, with expectation:

$$
\mathbb{E}[r_t|x_t \in D_t] = \mu^\top x_t
$$

(here $\mu$ is assumed constant over time).

Our regret:

$$
\mathbb{E}\left[\sum_t (\mu^\top x_t, a_t^*- \mu^\top x_t)|\mathcal{A}\right] \leq ??
$$

(where $x_t, a_t^*$ is the best decision at time $t$)
let's just run linUCB (or Thompson sampling)

Nothing really changes:
- $A_t$ and $b_t$ are the same updating rules
- now our decision is:

$$x_t = \arg\max_{x \in D_t} \max_{\nu \in B_t} \nu^T x$$

i.e.

$$x_t = \arg\max_{x \in D_t} \hat{\mu}_t^T x + (d \log t) \sqrt{x A_t^{-1} x}$$

Regret bound is still $O(d \sqrt{T})$. 
Acknowledgements

- https://sites.google.com/site/banditstutorial/
- http://www.yisongyue.com/courses/cs159/lectures/LinUCB.pdf