Case Study 1: Estimating Click Probabilities

SGD cont'd
AdaGrad

Machine Learning for Big Data
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Support/Resources

- Office Hours
  - Yao Lu: Tue 1:30-2:30, CSE 220
  - John Thickstun: Weds 4-5, CSE 220

Learning Problem for Click Prediction

- Prediction task: $X \rightarrow \{0, 1\}$ $\Pr(\text{click}=1 \mid X)$
- Features:
  $X = (\text{feats of page, ad, user, keyword})$
  $(\text{webpage5, ad7, user25, } \ldots)$
- Data:
  (x^i, y^i)
  - Batch:
    Fixed dataset $(x^1, y^1), \ldots, (x^N, y^N)$
  - Online: data as a stream
    user arrives at time t X_t $\rightarrow$ predict y $\rightarrow$ observe y_t
- Many approaches (e.g., logistic regression, SVMs, naïve Bayes, decision trees, boosting,…)
  - Focus on logistic regression; captures main concepts, ideas generalize to other approaches

Challenge 1: Complexity of computing gradients

- What's the cost of a gradient update step for LR???

O(d)

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})] \right\}$$

O(Nd) for this update.

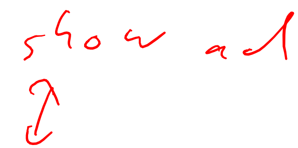
naively, O(Nd²) for all features.

with "caching" ∀ j $\left(\sum_j \vec{x}^j (y^j - \hat{y}^j) \right)$

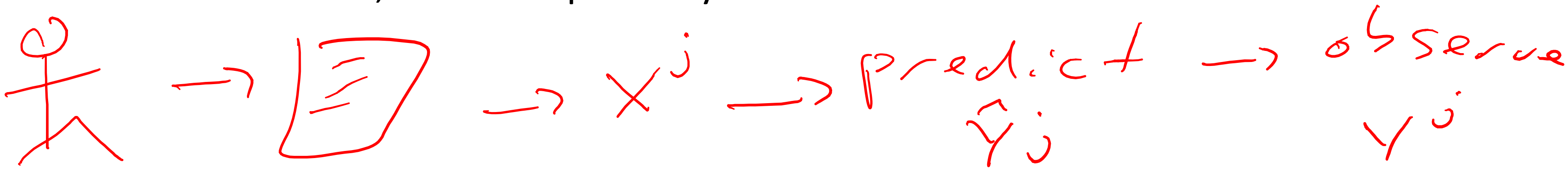
O(Nd) N is large (e.g. data)

Challenge 2: Data is streaming

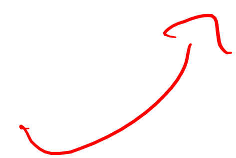
- Assumption thus far: **Batch data**
- But, click prediction is a streaming data task:
  - User enters query, and ad must be selected:



    - Observe $\mathbf{x}^j$, and must predict y^j



  - User either clicks or doesn't click on ad:



    - Label y^j is revealed afterwards
      - Google gets a reward if user clicks on ad
  - Weights must be updated for next time:

SGD: Stochastic Gradient Ascent (or Descent)

- "True" gradient: $\nabla \ell(\mathbf{w}) = E_{\mathbf{x}}\left[\nabla \ell(\mathbf{w}, \mathbf{x})\right]$

- Sample based approximation:

N determines approx quality

- What if we estimate gradient with just one sample???
  - Unbiased estimate of gradient
  - Very noisy!
  - Called stochastic gradient ascent (or descent)
    - Among many other names
  - VERY useful in practice!!!

Stochastic Gradient Ascent: General Case

- Given a stochastic function of parameters:
  - Want to find maximum
- Start from $\mathbf{w}^{(0)}$
- Repeat until convergence:
  - Get a sample data point $\mathbf{x}^t$
  - Update parameters:
- Works in the online learning setting!
- Complexity of each gradient step is constant in number of examples!
- In general, step size changes with iterations

Stochastic Gradient Ascent for Logistic Regression

- Logistic loss as a stochastic function:

$$E_{\mathbf{x}}\left[\ell(\mathbf{w},\mathbf{x})\right] = E_{\mathbf{x}}\left[\ln P(y|\mathbf{x},\mathbf{w}) - \frac{\lambda}{2}||\mathbf{w}||_2^2\right]$$

- Batch gradient ascent updates:

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta\left\{-\lambda w_i^{(t)} + \frac{1}{N}\sum_{j=1}^{N} x_i^{(j)}[y^{(j)} - P(Y=1|\mathbf{x}^{(j)},\mathbf{w}^{(t)})]\right\}$$

- Stochastic gradient ascent updates:
  - Online setting:

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta_t\left\{-\lambda w_i^{(t)} + x_i^{(t)}[y^{(t)} - P(Y=1|\mathbf{x}^{(t)},\mathbf{w}^{(t)})]\right\}$$

Convergence Rate of SGD

- **Theorem**:
  - (see Nemirovski et al '09 from readings)
  - Let *f* be a strongly convex stochastic function
  - Assume gradient of *f* is Lipschitz continuous and bounded
  - Then, for step sizes:
  - The expected loss decreases as O(1/t):

Convergence Rates for Gradient Descent/Ascent vs. SGD

- Number of Iterations to get to accuracy

$$\ell(\mathbf{w}^*) - \ell(\mathbf{w}) \leq \epsilon$$

- Gradient descent:
  - If func is strongly convex: $O(\ln(1/\epsilon))$ iterations
- Stochastic gradient descent:
  - If func is strongly convex: $O(1/\epsilon)$ iterations
- Seems exponentially worse, but much more subtle:
  - Total running time, e.g., for logistic regression:
    - Gradient descent:
    - SGD:
    - SGD can win when we have a lot of data
  - See readings for more details

Constrained SGD: Projected Gradient

- Consider an arbitrary restricted feature space $\mathbf{w} \in \mathcal{W}$

- Optimization objective:

- If $\mathbf{w} \in \mathcal{W}$, can use ***projected gradient*** for (sub)gradient descent

$$\mathbf{w}^{(t+1)} =$$

Motivating AdaGrad (Duchi, Hazan, Singer 2011)

- Assuming $\mathbf{w} \in \mathbb{R}^d$, standard stochastic (sub)gradient descent updates are of the form:

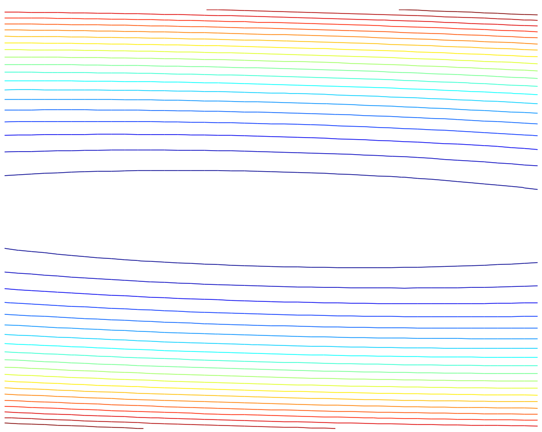
$$w_i^{(t+1)} \leftarrow w_i^{(t)} - \eta_t g_{t,i}$$

- Should all features share the same learning rate?

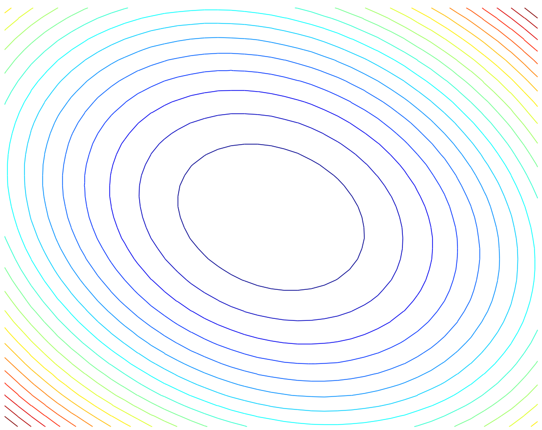
- Often have high-dimensional feature spaces
  - Many features are irrelevant
  - Rare features are often very informative

- Adagrad provides a feature-specific adaptive learning rate by incorporating knowledge of the geometry of past observations

Why Adapt to Geometry?



Hard



Nice

y_t	$x_{t,1}$	$x_{t,2}$	$x_{t,3}$
1	1	0	0
-1	.5	0	1
1	-.5	1	0
-1	0	0	0
1	.5	0	0
-1	1	0	0
1	-1	1	0
-1	-.5	0	1

Examples from Duchi et al. ISMP 2012 slides

1. Frequent, irrelevant
2. Infrequent, predictive
3. Infrequent, predictive

Not All Features are Created Equal

- Examples:

Text data:

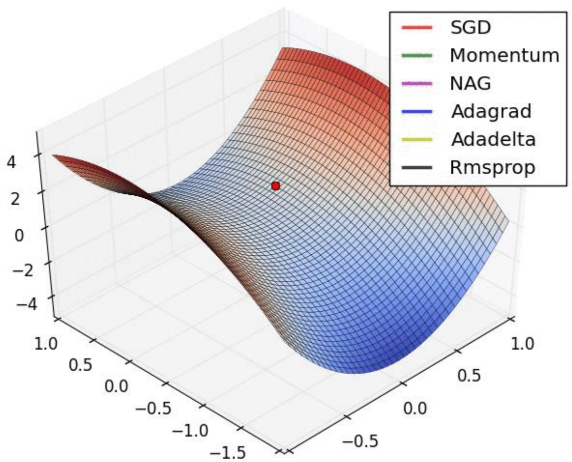
The most unsung birthday in American business and technological history this year may be the 50th anniversary of the Xerox 914 photocopier.[a]

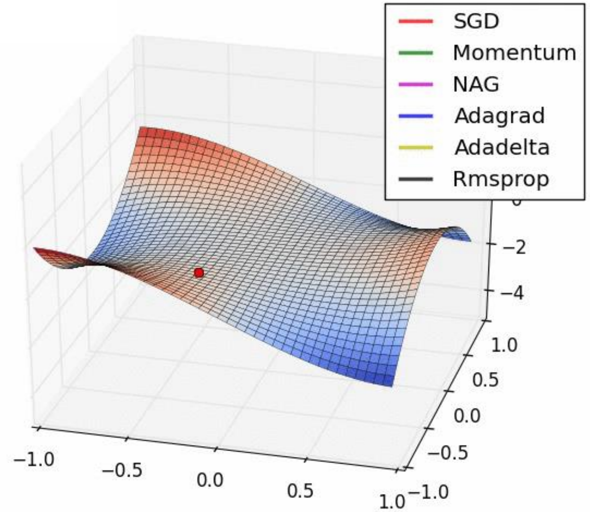
[a] *The Atlantic*, July/August 2010.

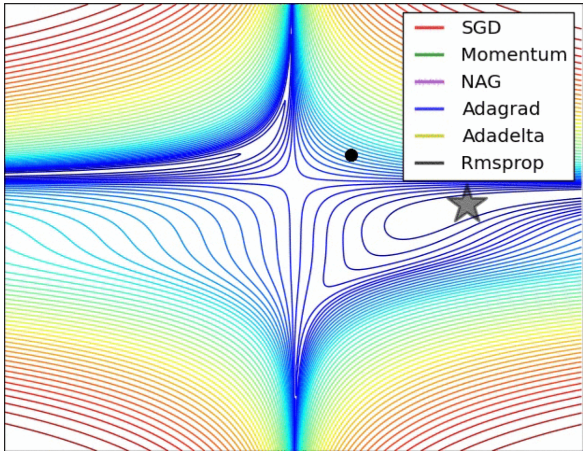
High-dimensional image features

Images from Duchi et al. ISMP 2012 slides

Visualizing Effect







Credit:
http://imgur.com/a/Hqolp

Regret Minimization

- How do we assess the performance of an online algorithm?

- Algorithm iteratively predicts $\mathbf{w}^{(t)}$
- Incur ***loss*** $\ell_t(\mathbf{w}^{(t)})$
- ***Regret***:
  What is the total incurred loss of algorithm relative to the best choice of $\mathbf{w}$ that could have been made ***retrospectively***

$$R(T) = \sum_{t=1}^{T} \ell_t(\mathbf{w}^{(t)}) - \inf_{\mathbf{w} \in \mathcal{W}} \sum_{t=1}^{T} \ell_t(\mathbf{w})$$

Regret Bounds for Standard SGD

- Standard projected gradient stochastic updates:

$$\mathbf{w}^{(t+1)} = \arg\min_{\mathbf{w}\in\mathcal{W}} ||\mathbf{w} - (\mathbf{w}^{(t)} - \eta g_t)||_2^2$$

- Standard regret bound:

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}^{(t)}) - \ell_t(\mathbf{w}^*) \leq \frac{1}{2\eta}||\mathbf{w}^{(1)} - \mathbf{w}^*||_2^2 + \frac{\eta}{2}\sum_{t=1}^{T}||g_t||_2^2$$

Projected Gradient using Mahalanobis

- Standard projected gradient stochastic updates:

$$\mathbf{w}^{(t+1)} = \arg\min_{\mathbf{w} \in \mathcal{W}} ||\mathbf{w} - (\mathbf{w}^{(t)} - \eta g_t)||_2^2$$

- What if instead of an L_2 metric for projection, we considered the ***Mahalanobis*** norm

$$\mathbf{w}^{(t+1)} = \arg\min_{\mathbf{w} \in \mathcal{W}} ||\mathbf{w} - (\mathbf{w}^{(t)} - \eta A^{-1} g_t)||_A^2$$

Mahalanobis Regret Bounds

$$\mathbf{w}^{(t+1)} = \arg\min_{\mathbf{w}\in\mathcal{W}} ||\mathbf{w} - (\mathbf{w}^{(t)} - \eta A^{-1} g_t)||_A^2$$

- **What *A* to choose?**
- Regret bound now:

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}^{(t)}) - \ell_t(\mathbf{w}^*) \leq \frac{1}{2\eta}||\mathbf{w}^{(1)} - \mathbf{w}^*||_2^2 + \frac{\eta}{2}\sum_{t=1}^{T} ||g_t||_{A^{-1}}^2$$

- What if we minimize upper bound on regret w.r.t. *A* in hindsight?

$$\min_A \sum_{t=1}^{T} g_t^T A^{-1} g_t$$

Mahalanobis Regret Minimization

- Objective:

$$\min_{A} \sum_{t=1}^{T} g_t^T A^{-1} g_t \qquad \text{subject to } A \succeq 0, \mathrm{tr}(A) \leq C$$

- Solution:

$$A = c \left(\sum_{t=1}^{T} g_t g_t^T \right)^{\frac{1}{2}}$$

For proof, see Appendix E, Lemma 15 of Duchi et al. 2011.
Uses "trace trick" and Lagrangian.

- *A* defines the norm of the metric space we should be operating in

AdaGrad Algorithm

- At time *t*, estimate optimal (sub)gradient modification *A* by

$$A_t = \left(\sum_{\tau=1}^{t} g_\tau g_\tau^T \right)^{\frac{1}{2}}$$

- For *d* large, A_t is computationally intensive to compute. Instead,

- Then, algorithm is a simple modification of normal updates:

$$\mathbf{w}^{(t+1)} = \arg\min_{\mathbf{w} \in \mathcal{W}} ||\mathbf{w} - (\mathbf{w}^{(t)} - \eta \mathrm{diag}(A_t)^{-1} g_t)||^2_{\mathrm{diag}(A_t)}$$

AdaGrad in Euclidean Space

- For $\mathcal{W} = \mathbb{R}^d$,
- For each feature dimension,

$$w_i^{(t+1)} \leftarrow w_i^{(t)} - \eta_{t,i} g_{t,i}$$

where

$$\eta_{t,i} =$$

- That is,

$$w_i^{(t+1)} \leftarrow w_i^{(t)} - \frac{\eta}{\sqrt{\sum_{\tau=1}^{t} g_{\tau,i}^2}} g_{t,i}$$

- Each feature dimension has it's own learning rate!
  - Adapts with *t*
  - Takes geometry of the past observations into account
  - Primary role of η is determining rate the first time a feature is encountered

AdaGrad Theoretical Guarantees

- AdaGrad regret bound:

$$R_\infty := \max_t ||\mathbf{w}^{(t)} - \mathbf{w}^*||_\infty$$

$$\sum_{t=1}^{T} \ell_t(\mathbf{w}^{(t)}) - \ell_t(\mathbf{w}^*) \leq 2R_\infty \sum_{i=1}^{d} ||g_{1:T,i}||_2$$

  - In stochastic setting:

$$\mathbb{E}\left[\ell\left(\frac{1}{T}\sum_{t=1}^{T} w^{(t)}\right)\right] - \ell(\mathbf{w}^*) \leq \frac{2R_\infty}{T} \sum_{i=1}^{d} \mathbb{E}[||g_{1:T,j}||_2]$$

- This really is used in practice!
- Many cool examples. Let's just examine one…

AdaGrad Theoretical Example

- Expect to out-perform when gradient vectors are *sparse*
- SVM hinge loss example:

$$\ell_t(\mathbf{w}) = [1 - y^t \langle \mathbf{x}^t, \mathbf{w} \rangle]_+$$

$$\mathbf{x}^t \in \{-1, 0, 1\}^d$$

- If $x_j^t \neq 0$ with probability $\propto j^{-\alpha}, \quad \alpha > 1$

$$\mathbb{E}\left[\ell\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}^{(t)}\right)\right] - \ell(\mathbf{w}^*) = \mathcal{O}\left(\frac{||\mathbf{w}^*||_\infty}{\sqrt{T}} \cdot \max\{\log d, d^{1-\alpha/2}\}\right)$$

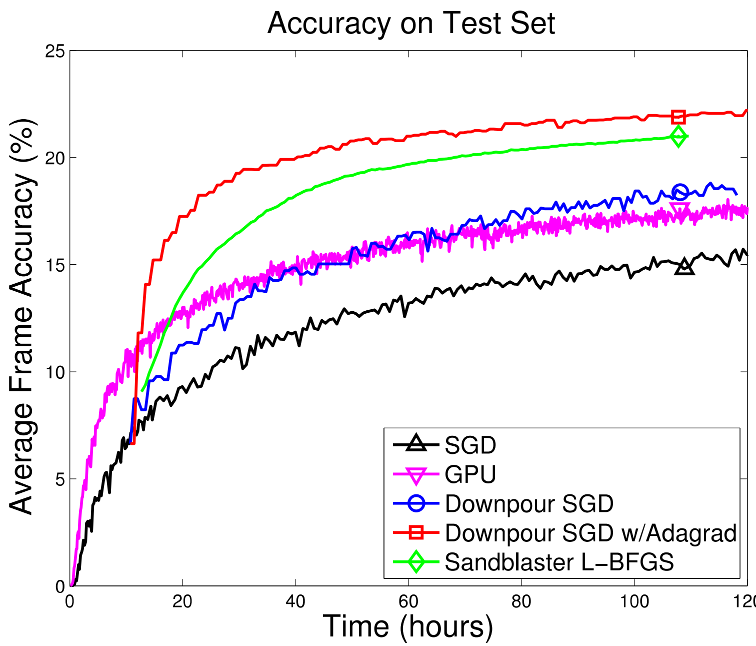
- Previously best known method: $\mathbb{E}\left[\ell\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}^{(t)}\right)\right] - \ell(\mathbf{w}^*) = \mathcal{O}\left(\frac{||\mathbf{w}^*||_\infty}{\sqrt{T}} \cdot \sqrt{d}\right)$

Neural Network Learning

- Very non-convex problem, but use SGD methods anyway

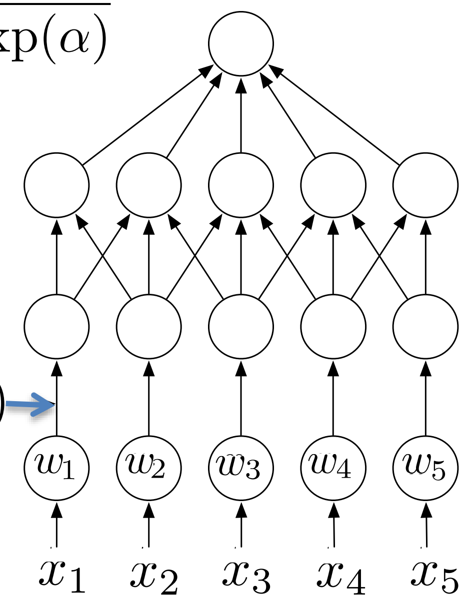
$$\ell(w,x) = \log(1+\exp(\langle[p(\langle w_1,x_1\rangle)\cdots p(\langle w_k,x_k\rangle)],x_0\rangle))$$

$$p(\alpha) = \frac{1}{1+\exp(\alpha)}$$



(Dean et al. 2012)

$p(\langle w_1, x_1\rangle)$



Distributed, $d = 1.7 \cdot 10^9$ parameters. SGD and AdaGrad use 80 machines (1000 cores), L-BFGS uses 800 (10000 cores)

Images from Duchi et al. ISMP 2012 slides

What you should know about Logistic Regression (LR) and Click Prediction

- Click prediction problem:
  - Estimate probability of clicking
  - Can be modeled as logistic regression
- Logistic regression model: Linear model
- Gradient ascent to optimize conditional likelihood
- Overfitting + regularization
- Regularized optimization
  - Convergence rates and stopping criterion
- Stochastic gradient ascent for large/streaming data
  - Convergence rates of SGD
- AdaGrad motivation, derivation, and algorithm