Online Learning
Perceptron Algorithm

Machine Learning – CSE546
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October 20, 2016

Challenge 1: Complexity of Computing Gradients

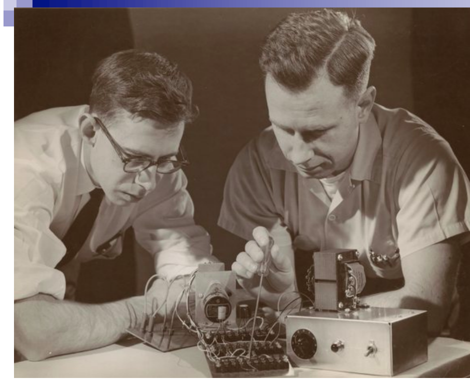
$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w}^{(t)})] \right\}$$

Challenge 2: Data is streaming

- Assumption thus far: **Batch data**
- But, e.g., in click prediction for ads is a streaming data task:
  - User enters query, and ad must be selected:
    - Observe $\mathbf{x}^j$, and must predict y^j
  - User either clicks or doesn't click on ad:
    - Label y^j is revealed afterwards
      - Google gets a reward if user clicks on ad
  - Weights must be updated for next time:

Online Learning Problem

- At each time step t:
  - Observe features of data point:
    - Note: many assumptions are possible, e.g., data is iid, data is adversarially chosen… details beyond scope of course
  - Make a prediction:
    - Note: many models are possible, we focus on linear models
    - *For simplicity, use vector notation*
  - Observe true label:
    - Note: other observation models are possible, e.g., we don't observe the label directly, but only a noisy version... Details beyond scope of course
  - Update model:




Rosenblatt 1957

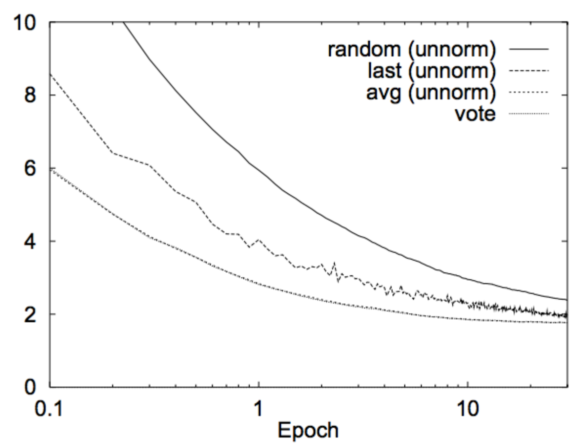
The Perceptron Algorithm [Rosenblatt '58, '62]

- Classification setting: y in {-1,+1}
- Linear model
  - Prediction:
- Training:
  - Initialize weight vector:
  - At each time step:
    - Observe features:
    - Make prediction:
    - Observe true class:
    - Update model:
      - If prediction is not equal to truth

Fundamental Practical Problem for All Online Learning Methods: **Which weight vector to report?**

- Perceptron prediction:
- Suppose you run online learning method and want to sell your learned weight vector… Which one do you sell???
- Last one?
-
-
-

Choice can make a huge difference!!



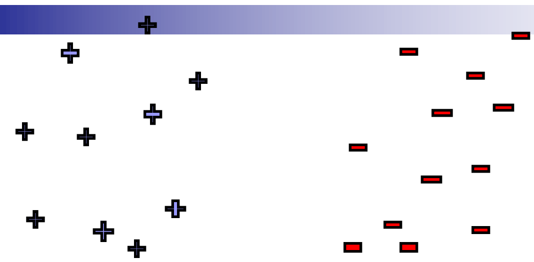
[Freund & Schapire ’99]

Mistake Bounds

- Algorithm "pays" every time it makes a mistake:

- How many mistakes is it going to make?

Linear Separability: More formally, Using Margin



- Data linearly separable, if there exists
  - a vector
  - a margin
- Such that

Perceptron Analysis: Linearly Separable Case

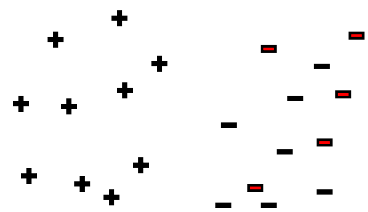
- Theorem [Block, Novikoff]:
  - Given a sequence of labeled examples:
  - Each feature vector has bounded norm:
  - If dataset is linearly separable:
- Then the number of mistakes made by the online perceptron on any such sequence is bounded by

Perceptron Proof for Linearly Separable case

- Every time we make a mistake, we get gamma closer to w^*:
  - Mistake at time t: $w^{(t+1)} = w^{(t)} + y^{(t)} x^{(t)}$
  - Taking dot product with w^*:
  - Thus after m mistakes:
- Similarly, norm of $w^{(t+1)}$ doesn't grow too fast:
  - $||\mathbf{w}^{(t+1)}||^2 = ||\mathbf{w}^{(t)}||^2 + 2y^{(t)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(t)}) + ||\mathbf{x}^{(t)}||^2$
  - Thus, after m mistakes:
- Putting all together:

Beyond Linearly Separable Case

- Perceptron algorithm is super cool!
  - No assumption about data distribution!
    - Could be generated by an oblivious adversary, no need to be iid
  - Makes a fixed number of mistakes, and it's done for ever!
    - Even if you see infinite data
- However, real world not linearly separable
  - Can't expect never to make mistakes again
  - Analysis extends to non-linearly separable case
  - Very similar bound, see Freund & Schapire
  - Converges, but ultimately may not give good accuracy (make many many many mistakes)



What you need to know

- Notion of online learning
- Perceptron algorithm
- Mistake bounds and proof
- In online learning, report averaged weights at the end

What's the Perceptron Optimizing?

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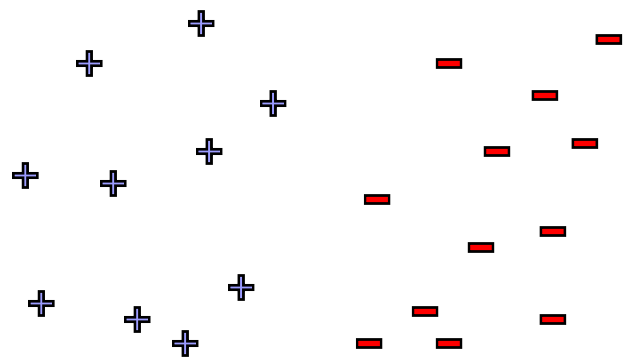
What is the Perceptron Doing???

- When we discussed logistic regression:
  - Started from maximizing conditional log-likelihood
- When we discussed the Perceptron:
  - Started from description of an algorithm
- What is the Perceptron optimizing????

Perceptron Prediction: Margin of Confidence



Hinge Loss

- Perceptron prediction:
- Makes a mistake when:
- Hinge loss (same as maximizing the margin used by SVMs)

Minimizing hinge loss in Batch Setting

- Given a dataset:
- Minimize average hinge loss:
- How do we compute the gradient?

Subgradients of Convex Functions

- Gradients lower bound convex functions:
- Gradients are unique at **w** iff function differentiable at **w**
- Subgradients: Generalize gradients to non-differentiable points:
  - Any plane that lower bounds function:

Subgradient of Hinge

- Hinge loss:

- Subgradient of hinge loss:
  - If $y^{(t)} (w.\mathbf{x}^{(t)}) > 0$:
  - If $y^{(t)} (w.\mathbf{x}^{(t)}) < 0$:
  - If $y^{(t)} (w.\mathbf{x}^{(t)}) = 0$:
  - In one line:

Subgradient Descent for Hinge Minimization

- Given data:

- Want to minimize:

- Subgradient descent works the same as gradient descent:
  - But if there are multiple subgradients at a point, just pick (any) one:

Perceptron Revisited

- Perceptron update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \mathbb{1}\left[y^{(t)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(t)}) \leq 0\right] y^{(t)}\mathbf{x}^{(t)}$$

- Batch hinge minimization update:

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + \eta \frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{1}\left[y^{(i)}(\mathbf{w}^{(t)} \cdot \mathbf{x}^{(i)}) \leq 0\right] y^{(i)}\mathbf{x}^{(i)} \right\}$$

- Difference?

What you need to know

- Perceptron is optimizing hinge loss
- Subgradients and hinge loss
- (Sub)gradient decent for hinge objective