Overfitting

Machine Learning – CSE546

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October 6, 2016

Announcements:

- HW1 posted

- Today:
  - Review: bias variance
  - Overfitting, overfitting overfitting
  - Ridge regression
  - Checking for overfitting: cross validation

Review:
Bias-Variance Tradeoff

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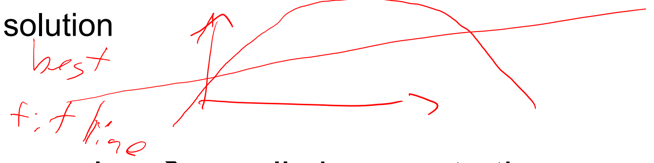
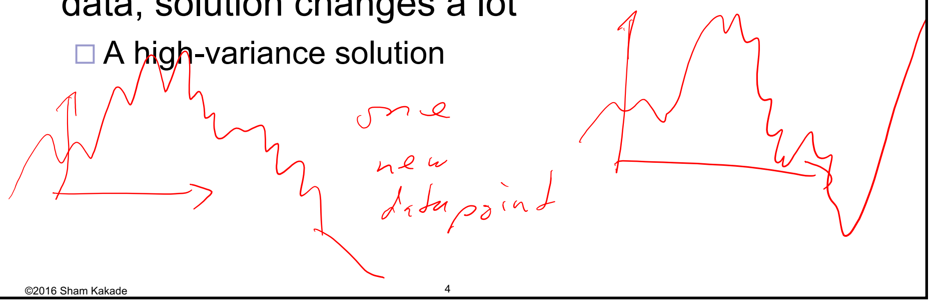
Bias-Variance tradeoff – Intuition

- Model too "simple" ➔ does not fit the data well
  - A biased solution



- Model too complex ➔ small changes to the data, solution changes a lot
  - A high-variance solution



(Squared) Bias of learner

$D \longrightarrow$ learn h_D

- Given dataset *D* with *N* samples, learn function $h_D(x)$
- If you sample a different dataset *D'* with *N* samples, you will learn different $h_D{}'(x)$
- **Expected hypothesis**: $E_D[h_D(x) | x]$ $E_D[h_D(x) | X]$

  $\bar{h}_N =$

- **Bias:** difference between what you expect to learn and truth
  - Measures how well you expect to represent true solution
  - Decreases with more complex model
  - Bias^2 at one point *x*: $\rightarrow (f(x) - \bar{h}_N(x))^2$
  - Average Bias^2: $E_X[(f(x) - \bar{h}_N(x))^2]$

Variance of learner

$D \longrightarrow h_D.$

- Given dataset *D* with *N* samples, learn function $h_D(x)$
- If you sample a different dataset *D'* with *N* samples, you will learn different $h_D{}'(x)$
- **Variance:** difference between what you expect to learn and what you learn from a particular dataset
  - Measures how sensitive learner is to specific dataset
  - Decreases with simpler model
  - Variance at one point *x*: $\rightarrow E_D[(h_D(x) - \bar{h}_N(x))^2 | X]$
  - Average variance: $E_X E_D[(h_D(x) - \bar{h}_N(x))^2]$.

Training set error

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_j \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

- Given a dataset (Training data)
- Choose a loss function
  - e.g., squared error (L_2) for regression
- **Training set error:** For a particular set of parameters, loss function on training data:

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

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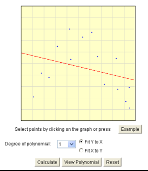
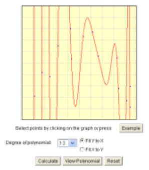
Training set error as a function of model complexity

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$


Select points by clicking on the graph or press Example

Degree of polynomial: 1 [✓ Fit Y to X] Fit X to Y


Select points by clicking on the graph or press Example

Degree of polynomial: 13 [✓ Fit Y to X] Fit X to Y

Prediction error

Select points by clicking on the graph or press Example

Degree of polynomial: 13 [✓ Fit Y to X] Fit X to Y

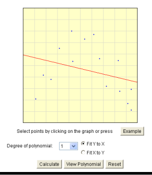
- Training set error can be poor measure of "quality" of solution
- **Prediction error:** We really care about error over all possible input points, not just training data:

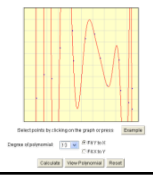
$$\begin{aligned} error_{true}(\mathbf{w}) &= E_{\mathbf{x}} \left[\left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 \right] \\ &= \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x} \end{aligned}$$

Prediction error as a function of model complexity

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$


Select points by clicking on the graph or press
Degree of polynomial: 1
[✓ Fit Y to X] Fit X to Y


Select points by clicking on the graph or press
Degree of polynomial: 13
[✓ Fit Y to X] Fit X to Y

Computing prediction error

- Computing prediction
  - Hard integral
  - May not know t(**x**) for every **x**

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

- Monte Carlo integration (sampling approximation)
  - Sample a set of i.i.d. points {$\mathbf{x}_1,\ldots,\mathbf{x}_M$} from p(**x**)
  - Approximate integral with sample average

$$error_{true}(\mathbf{w}) \approx \frac{1}{M} \sum_{j=1}^{M} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

Why training set error doesn't approximate prediction error?

- Sampling approximation of prediction error:

$$error_{true}(\mathbf{w}) \approx \frac{1}{M}\sum_{j=1}^{M}\left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j)\right)^2$$

- Training error :

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}}\sum_{j=1}^{N_{train}}\left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j)\right)^2$$

- Very similar equations!!!
  - Why is training set a bad measure of prediction error???

Why training set error doesn't approximate prediction error?

Because you cheated!!!

Training error good estimate for a single **w,**
But you optimized **w** with respect to the training error,
and found **w** that is good for this set of samples

Training error is a (optimistically) biased estimate of prediction error

- Very similar equations!!!
  - Why is training set a bad measure of prediction error???

Test set error

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_j \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

- Given a dataset, **randomly** split it into two parts:
  - Training data – {$\mathbf{x}_1, \ldots, \mathbf{x}_{Ntrain}$}
  - Test data – {$\mathbf{x}_1, \ldots, \mathbf{x}_{Ntest}$}
- Use training data to optimize parameters **w**
- **Test set error:** For the ***final output*** $\hat{\mathbf{w}}$, evaluate the error using:

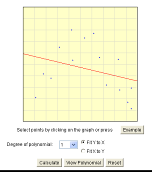
$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

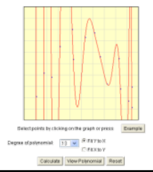
Test set error as a function of model complexity

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$





Overfitting

- **Overfitting:** a learning algorithm overfits the training data if it outputs a solution **w** when there exists another solution **w'** such that:

$$[error_{train}(\mathrm{w}) < error_{train}(\mathrm{w}')] \wedge [error_{true}(\mathrm{w}') < error_{true}(\mathrm{w})]$$

How many points to I use for training/testing?

- Very hard question to answer!
  - Too few training points, learned **w** is bad
  - Too few test points, you never know if you reached a good solution
- Bounds, such as Hoeffding's inequality can help:

$$P(|\,\widehat{\theta} - \theta^* \,| \geq \epsilon) \;\leq\; 2e^{-2N\epsilon^2}$$

- More on this later this quarter, but still hard to answer
- Typically:
  - If you have a reasonable amount of data, pick test set "large enough" for a "reasonable" estimate of error, and use the rest for learning
  - If you have little data, then you need to pull out the big guns…
    - e.g., bootstrapping

Error estimators

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

Error as a function of number of training examples for a fixed model complexity

Degree of polynomial: 13
[✓ Fit Y to X]
Fit X to Y

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

little data $\longrightarrow$ infinite data

Error estimators

Be careful!!!

Test set only unbiased if you never never ever ever do any any any any learning on the test data

For example, if you use the test set to select the degree of the polynomial… no longer unbiased!!! (We will address this problem later)

err

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$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

What you need to know

- True error, training error, test error
  - Never learn on the test data
  - Never learn on the test data
  - Never learn on the test data
  - Never learn on the test data
  - Never learn on the test data
- Overfitting

Regularization

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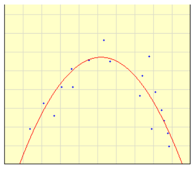
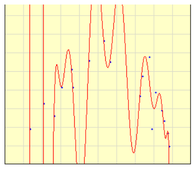
Regularization in Linear Regression

- Overfitting usually leads to very large parameter choices, e.g.:

$-2.2 + 3.1\ X - 0.30\ X^2$ $-1.1 + 4{,}700{,}910.7\ X - 8{,}585{,}638.4\ X^2 + \ldots$

- ***Regularized*** or ***penalized*** regression aims to impose a "complexity" penalty by penalizing large weights
  - "Shrinkage" method

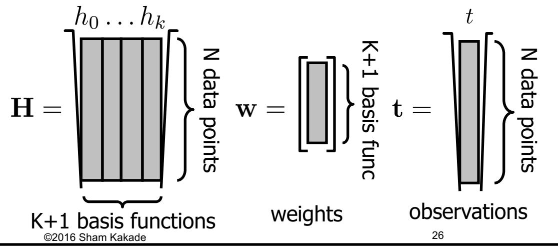
Ridge Regression

- Aleviating issues with overfitting:
- New objective:

Ridge Regression in Matrix Notation

$$\hat{\mathbf{w}}_{ridge} = \arg\min_{w} \sum_{j=1}^{N} \left(t(x_j) - (w_0 + \sum_{i=1}^{k} w_i h_i(x_j)) \right)^2 + \lambda \sum_{i=1}^{k} w_i^2$$

$$= \arg\min_{\mathbf{w}} \underbrace{(\mathbf{Hw} - \mathbf{t})^T (\mathbf{Hw} - \mathbf{t})}_{\text{residual error}} + \lambda\ \mathbf{w}^T I_{0+k} \mathbf{w}$$



Minimizing the Ridge Regression Objective

$$\hat{\mathbf{w}}_{ridge} = \arg\min_w \sum_{j=1}^N \left(t(x_j) - (w_0 + \sum_{i=1}^k w_i h_i(x_j)) \right)^2 + \lambda \sum_{i=1}^k w_i^2$$
$$= (H\mathbf{w} - \mathbf{t})^T (H\mathbf{w} - \mathbf{t}) + \ \lambda\ \mathbf{w}^T I_{0+k} \mathbf{w}$$

Shrinkage Properties

$$\hat{\mathbf{w}}_{ridge} = (H^T H + \lambda\ I_{0+k})^{-1} H^T \mathbf{t}$$

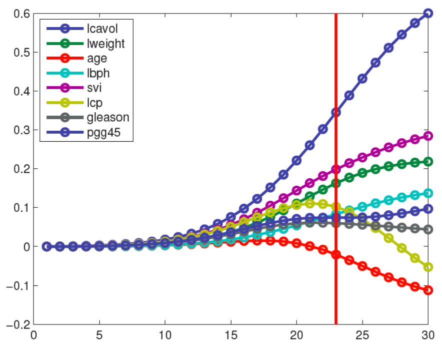
- If orthonormal features/basis: $H^T H = I$

Ridge Regression: Effect of Regularization

$$\hat{\mathbf{w}}_{ridge} = \arg\min_w \sum_{j=1}^{N} \left(t(x_j) - (w_0 + \sum_{i=1}^{k} w_i h_i(x_j)) \right)^2 + \lambda \sum_{i=1}^{k} w_i^2$$

- Solution is indexed by the regularization parameter λ
- Larger λ
- Smaller λ
- As λ → 0
- As λ →∞

Ridge Coefficient Path



From
Kevin Murphy
textbook

- Typical approach: select λ using cross validation, more on this later in the quarter

Error as a function of regularization parameter for a fixed model complexity

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

$\lambda=\infty$ $\longrightarrow$ $\lambda=0$

What you need to know…

- Regularization
  - Penalizes for complex models
- Ridge regression
  - L_2 penalized least-squares regression
  - Regularization parameter trades off model complexity with training error

Cross-Validation

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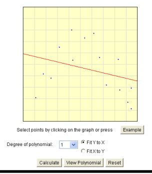
Test set error as a function of model complexity

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

$$error_{test}(\mathbf{w}) = \frac{1}{N_{test}} \sum_{j=1}^{N_{test}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$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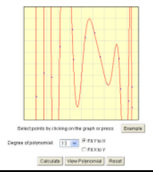

Select points by clicking on the graph or press
Degree of polynomial: 13 [✓ Fit Y to X] Fit X to Y

How… How… How???????

- How do we pick the regularization constant λ…
  - And all other constants in ML, 'cause one thing ML doesn't lack is constants to tune… ☹
- We could use the test data, but…

(LOO) Leave-one-out cross validation

- Consider a **validation set with 1 example**:
  - *D* – training data
  - *D*\j – training data with *j* th data point moved to validation set
- **Learn classifier $h_{D\backslash j}$ with *D*\j dataset**
- **Estimate true error** as squared error on predicting t($\mathbf{x}_j$):
  - Unbiased estimate of $\text{error}_{\text{true}}(h_{D\backslash j})$!
  - Seems really bad estimator, but wait!
- **LOO cross validation**: Average over all data points *j*:
  - **For each data point you leave out, learn a new classifier $h_{D\backslash j}$**
  - **Estimate error** as:

$$error_{LOO} = \frac{1}{N}\sum_{j=1}^{N}\left(t(\mathbf{x}_j) - h_{\mathcal{D}\backslash j}(\mathbf{x}_j)\right)^2$$

LOO cross validation is (almost) unbiased estimate of true error of h_D!

- When computing **LOOCV error, we only use *N-1* data points**
  - So it's not estimate of true error of learning with *N* data points!
  - Usually pessimistic, though – learning with less data typically gives worse answer
- **LOO is almost unbiased**!
- **Great news!**
  - **Use LOO error for model selection!!!**
  - **E.g., picking λ**

Using LOO to Pick λ

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x})d\mathbf{x}$$

$$error_{LOO} = \frac{1}{N} \sum_{j=1}^{N} \left(t(\mathbf{x}_j) - h_{\mathcal{D}\backslash j}(\mathbf{x}_j) \right)^2$$

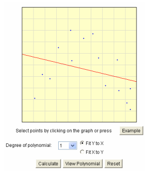
λ=∞ $\longrightarrow$ λ=0

Using LOO error for model selection

$$error_{train}(\mathbf{w}) = \frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$$

$$error_{true}(\mathbf{w}) = \int_{\mathbf{x}} \left(t(\mathbf{x}) - \sum_i w_i h_i(\mathbf{x}) \right)^2 p(\mathbf{x}) d\mathbf{x}$$

$$error_{LOO} = \frac{1}{N} \sum_{j=1}^{N} \left(t(\mathbf{x}_j) - h_{\mathcal{D}\setminus j}(\mathbf{x}_j) \right)^2$$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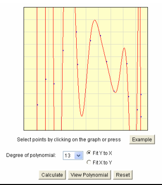
Degree of polynomial: 1

[✓ Fit Y to X] Fit X to Y


Select points by clicking on the graph or press

Degree of polynomial: 13

[✓ Fit Y to X] Fit X to Y

Computational cost of LOO

- Suppose you have 100,000 data points
- You implemented a great version of your learning algorithm
  - Learns in only 1 second
- Computing LOO will take about 1 day!!!
  - If you have to do for each choice of basis functions, it will take fooooooreeeve'!!!
- Solution 1: Preferred, but not usually possible
  - Find a cool trick to compute LOO (e.g., see homework)

Solution 2 to complexity of computing LOO: (More typical) **Use *k*-fold cross validation**

- Randomly **divide training data into *k* equal parts**
  - *D₁,…,D_k*
- For each *i*
  - **Learn classifier $h_{D \backslash D_i}$ using data point not in D_i**
  - **Estimate error of $h_{D \backslash D_i}$ on validation set D_i:**

$$error_{\mathcal{D}_i} = \frac{k}{N} \sum_{\mathbf{x}_j \in \mathcal{D}_i} \left(t(\mathbf{x}_j) - h_{\mathcal{D} \backslash \mathcal{D}_i}(\mathbf{x}_j)\right)^2$$

- ***k*-fold cross validation error is average** over data splits:

$$error_{k-fold} = \frac{1}{k} \sum_{i=1}^{k} error_{\mathcal{D}_i}$$

- *k*-fold cross validation properties:
  - **Much faster to compute** than LOO
  - **More (pessimistically) biased** – using much less data, only *m(k-1)/k*
  - Usually, **k = 10** ☺

What you need to know…

- Never ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever ever train on the test data
- Use cross-validation to choose magic parameters such as λ
- Leave-one-out is the best you can do, but sometimes too slow
  - In that case, use k-fold cross-validation