Markov Decision Processes (MDPs)

Machine Learning – CSE546

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Reinforcement Learning

training by feedback

Learning to act

- Reinforcement learning
- An agent
  - Makes sensor observations
  - Must select action
  - Receives rewards
    - positive for "good" states
    - negative for "bad" states

[Ng et al. '05]

Markov Decision Process (MDP) Representation

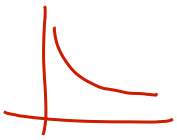
- State space:
  - Joint state **x** of entire system
- Action space:
  - Joint action **a**= {a_1,…, a_n} for all agents
- Reward function:
  - Total reward R(**x**,**a**)
    - sometimes reward can depend on action
- Transition model:
  - Dynamics of the entire system P(**x**'|**x**,**a**)

Discount Factors

People in economics and probabilistic decision-making do this all the time.

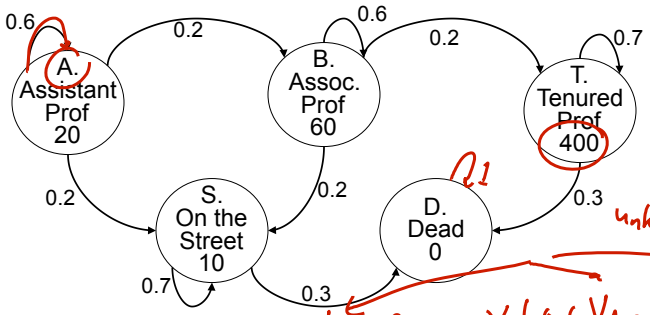
The "Discounted sum of future rewards" using discount factor γ" is

(reward now) +
γ (reward in 1 time step) +
γ^2 (reward in 2 time steps) +
γ^3 (reward in 3 time steps) +
:
: (infinite sum)



The Academic Life

Assume Discount Factor $\gamma = 0.9$



Define:

V_A = Expected discounted future rewards starting in state A

V_B = Expected discounted future rewards starting in state B

V_T = " " " " " " " T

V_S = " " " " " " " S

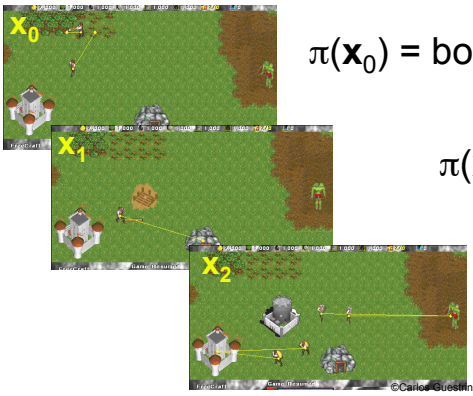
V_D = " " " " " " " D

How do we compute V_A, V_B, V_T, V_S, V_D ?

Policy

Policy: $\pi(\mathbf{x}) = \mathbf{a}$

At state **x**, action **a** for all agents



$\pi(\mathbf{x}_0)$ = both peasants get wood

$\pi(\mathbf{x}_1)$ = one peasant builds barrack, other gets gold

$\pi(\mathbf{x}_2)$ = peasants get gold, footmen attack

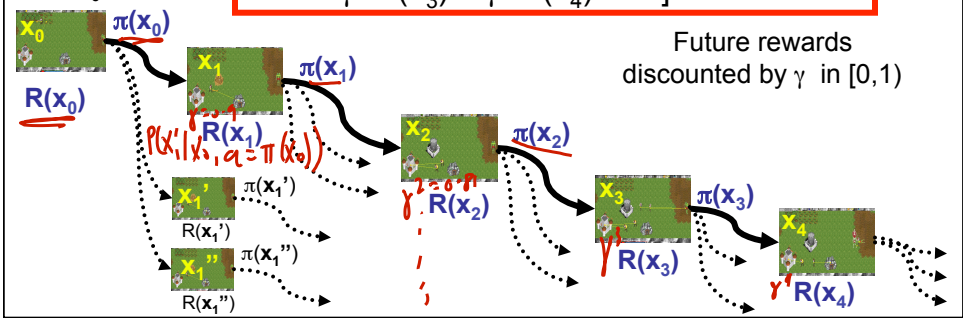
Value of Policy

Value: $V_\pi(\mathbf{x})$

Expected long-term reward starting from **x**

$$V_\pi(\mathbf{x}_0) = \mathbf{E}_\pi[R(\mathbf{x}_0) + \gamma\, R(\mathbf{x}_1) + \gamma^2\, R(\mathbf{x}_2) + \gamma^3\, R(\mathbf{x}_3) + \gamma^4\, R(\mathbf{x}_4) + \ldots]$$

Start from $\mathbf{x}_0$



Future rewards discounted by γ in [0,1)

Computing the value of a policy

$V_\pi(\mathbf{x_0}) = \mathbf{E}_\pi[R(\mathbf{x}_0) + \gamma\, R(\mathbf{x}_1) + \gamma^2\, R(\mathbf{x}_2) + \gamma^3\, R(\mathbf{x}_3) + \gamma^4\, R(\mathbf{x}_4) + \ldots]$

- Discounted value of a state:
  - value of starting from x_0 and continuing with policy π from then on

$$V_\pi(x_0) = E_\pi[R(x_0) + \gamma R(x_1) + \gamma^2 R(x_2) + \gamma^3 R(x_3) + \cdots]$$
$$= E_\pi[\sum_{t=0}^{\infty} \gamma^t R(x_t)] = E_\pi[R(x_0) + \gamma \sum_{t=1}^{\infty} \gamma^{(t-1)} R(x_t)]$$

$V_\pi(x_1)$

- A recursion!

$$= R(x_0) + \gamma \sum_{x_1} P(x_1 | x_0, a=\pi(x_0)) V_\pi(x_1)$$

Simple approach for computing the value of a policy: Iteratively

$$V_\pi(x) = R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- Can solve using a simple convergent iterative approach: (a.k.a. dynamic programming)
  - Start with some guess V^0 ← value for each state ← e.g. 0 or $V^0(x) = R(x)$
  - Iteratively say:
    - $V_\pi^{t+1}(x) \leftarrow R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi^t(x')$ → iterate
  - Stop when $||V_{t+1} - V_t||_\infty < \varepsilon$
    - means that $||V_\pi - V_{t+1}||_\infty < \varepsilon/(1-\gamma)$

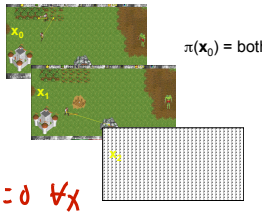
But we want to learn a **Policy**

- So far, told you how good a policy is…
- But how can we choose the best policy???
- Suppose there was only one time step:
  - world is about to end!!!
  - select action that maximizes reward!

$V^{t+1}(x) = 0 \ \forall x$

$V(x) = \max_a R(x,a)$

Policy: $\pi(\mathbf{x}) = \mathbf{a}$ → At state **x**, action **a** for all agents



$\pi(\mathbf{x}_0)$ = both peasants get wood

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Unrolling the recursion

- Choose actions that lead to best value in the long run
  - Optimal value policy achieves optimal value V^*

$$V^*(x_0) = \max_{a_0} R(x_0, a_0) + \gamma E_{a_0}[\max_{a_1} R(x_1) + \gamma^2 E_{a_1}[\max_{a_2} R(x_2) + \cdots]]$$

$V^*(x_0) = \max_a R(x,a) + \gamma \sum_{x_1} P(x_1 | x_0, a) V^*(x_1)$

for each x_0

n - states
k - actions

n unknowns → value of each state

n - equations

⇒ hard because non linear

max

Bellman equation

- Evaluating policy π:

$$V_\pi(x) = R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- Computing the optimal value V^* - Bellman equation

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x}, \mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x}, \mathbf{a}) V^*(\mathbf{x}')$$

Solving an MDP

Solve Bellman equation → Optimal value $V^*(\mathbf{x})$ → Optimal policy $\pi^*(\mathbf{x})$

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x}, \mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x}, \mathbf{a}) V^*(\mathbf{x}')$$

Bellman equation is non-linear!!!

Many algorithms solve the Bellman equations:

- Policy iteration [Howard '60, Bellman '57]
- Value iteration [Bellman '57]
- Linear programming [Manne '60]
- …

Value iteration (a.k.a. dynamic programming) – the simplest of all

$$V^*(x) = R(x,a) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V^*(x')$$

- Start with some guess V^0
- Iteratively say:
  - $V^{t+1}(x) \leftarrow \max_a R(x,a) + \gamma \sum_{x'} P(x' \mid x, a) V^t(x')$
- Stop when $||V_{t+1} - V_t||_\infty < \varepsilon$
  - means that $||V^* - V_{t+1}||_\infty < \varepsilon/(1-\gamma)$

Optimal Long-term Plan

Optimal value function $V^*(\mathbf{x})$ → Optimal Policy: $\pi^*(\mathbf{x})$

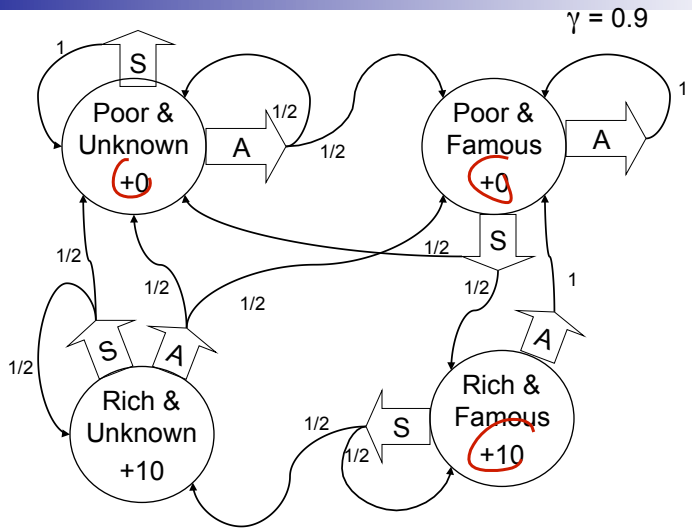
Optimal policy:

$$\pi^*(\mathbf{x}) = \arg\max_a R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

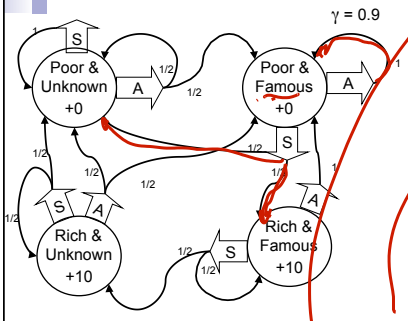
A simple example

You run a startup company.

In every state you must choose between Saving money or Advertising.



Let's compute $V_t(x)$ for our example

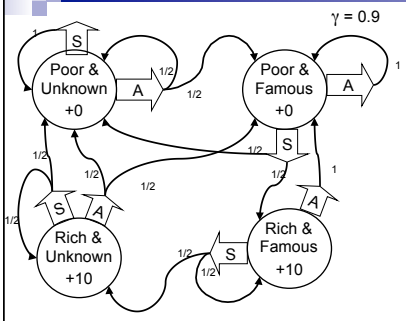


t	V^t(PU)	V^t(PF)	V^t(RU)	V^t(RF)
1	0	0	10	10
2		4.5		
3				
4				
5				
6				

$$V^2(PF) = \max_a \begin{cases} A: & 0 + 0.9\,V^1(PF) \\ S: & 0 + 0.9\,(.5\,V^1(PU) + 0.5\,V^1(RF)) \end{cases}$$

$$V^{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^t(\mathbf{x}')$$

Let's compute $V_t(x)$ for our example



t	V^t(PU)	V^t(PF)	V^t(RU)	V^t(RF)
1	0	0	10	10
2	0	4.5	14.5	19
3	2.03	9.46	17.44	25.08
4	5.17	13.61	20.17	29.13
5	8.45	16.91	22.88	32.19
6	11.41	19.62	25.43	34.78
∞	31.59	38.60	44.02	54.02

$$V^{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^{t}(\mathbf{x}')$$

What you need to know

- What's a Markov decision process
  - state, actions, transitions, rewards
  - a policy
  - value function for a policy
    - computing V_π
- Optimal value function and optimal policy
  - Bellman equation
- Solving Bellman equation
  - with value iteration, policy iteration and linear programming

Acknowledgment

- This lecture contains some material from Andrew Moore's excellent collection of ML tutorials:
  - http://www.cs.cmu.edu/~awm/tutorials

Rewards & probs are unknown

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The Reinforcement Learning task

World: You are in state 34.
Your immediate reward is 3. You have possible 3 actions.

Robot: I'll take action 2.
World: You are in state 77.
Your immediate reward is -7. You have possible 2 actions.

Robot: I'll take action 1.
World: You're in state 34 (again).
Your immediate reward is 3. You have possible 3 actions.

Formalizing the (online) reinforcement learning problem

- Given a set of states **X** and actions **A**
  - in some versions of the problem size of **X** and **A** unknown
- Interact with world at each time step *t*:
  - world gives state $\mathbf{x}_t$ and reward r_t
  - you give next action $\mathbf{a}_t$
- **Goal**: (quickly) learn policy that (approximately) maximizes long-term expected discounted reward

The "Credit Assignment" Problem

I'm in state 43,	reward = 0,	action = 2
" " " 39,	" = 0,	" = 4
" " " 22,	" = 0,	" = 1
" " " 21,	" = 0,	" = 1
" " " 21,	" = 0,	" = 1
" " " 13,	" = 0,	" = 2
" " " 54,	" = 0,	" = 2
" " " 26,	" = 100,	

Yippee! I got to a state with a big reward! But which of my actions along the way actually helped me get there??

This is the Credit Assignment problem.

Exploration-Exploitation tradeoff

- You have visited part of the state space and found a reward of 100
  - is this the best I can hope for???
- **Exploitation**: should I stick with what I know and find a good policy w.r.t. this knowledge?
  - at the risk of missing out on some large reward somewhere
- **Exploration**: should I look for a region with more reward?
  - at the risk of wasting my time or collecting a lot of negative reward

Rmax – A model-based approach

Given a dataset – learn model

Given data, learn (MDP) Representation:

- Dataset: $x_t, a_t \rightarrow r_t, x_{t+1}$
- Learn reward function:
  - R(**x**,**a**) $R(x_t, a_t) = r_t$
- Learn transition model:
  - P(**x**'|**x**,**a**) next, current state, action

MLE $= \frac{Count(x_{t+1}=x', x_t=x, A_t=a)}{Count(x_t=x, A_t=a)}$

Planning with insufficient information

- Model-based approach:
  - estimate R(**x**,**a**) & P(**x**'|**x**,**a**)
  - obtain policy by value or policy iteration, or linear programming
  - No credit assignment problem!
    - learning model, planning algorithm takes care of "assigning" credit
- What do you plug in when you don't have enough information about a state?
  - don't reward at a particular state
    - plug in 0?
    - plug in smallest reward (R_{min})? ← pessimistic
    - plug in largest reward (R_{max})? ← optimistic

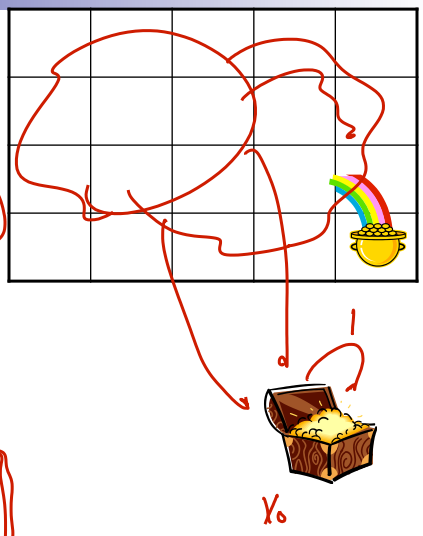
    $\frac{R_{max}+R_{min}}{2}$ ← avg
  - don't know a particular transition probability?

A surprisingly simple approach for model based RL – The Rmax algorithm [Brafman & Tennenholtz]

- **Optimism in the face of uncertainty!!!!**
  - heuristic shown to be useful long before theory was done (e.g., Kaelbling '90)
- If you don't know reward for a particular state-action pair, set it to R_{max}!!!
- If you don't know the transition probabilities P(**x'**|**x**,**a**) from some some state action pair **x**,**a** assume you go to **a magic, fairytale** new state $\mathbf{x}_0$!!!
  - $R(\mathbf{x}_0,\mathbf{a}) = R_{max}$
  - $P(\mathbf{x}_0|\mathbf{x}_0,\mathbf{a}) = 1$

Understanding R_{max}

- With R_{max} you either:
  - **explore** – visit a state-action pair you don't know much about
    - because it seems to have lots of potential
  - **exploit** – spend all your time on known states
    - even if unknown states were amazingly good, it's not worth it
- Note: you never know if you are exploring or exploiting!!!



Implicit Exploration-Exploitation Lemma

- **Lemma**: every T time steps, either:
  - **Exploits**: achieves near-optimal reward for these T-steps, or
  - **Explores**: with high probability, the agent visits an unknown state-action pair
    - learns a little about an unknown state
  - T is related to *mixing time* of Markov chain defined by MDP
    - time it takes to (approximately) forget where you started

act
optimistic → optimally
optimistic → learn something
don't know which is happening

The Rmax algorithm

- **Initialization**:
  - Add state $\mathbf{x}_0$ to MDP
  - $R(\mathbf{x},\mathbf{a}) = R_{max}$, $\forall \mathbf{x},\mathbf{a}$
  - $P(\mathbf{x}_0|\mathbf{x},\mathbf{a}) = 1$, $\forall \mathbf{x},\mathbf{a}$
  - all states (except for $\mathbf{x}_0$) are **unknown**
- Repeat
  - obtain policy for current MDP and Execute policy
  - for any visited state-action pair, set reward function to appropriate value
  - if visited some state-action pair $\mathbf{x},\mathbf{a}$ enough times to estimate $P(\mathbf{x}'|\mathbf{x},\mathbf{a})$
    - update transition probs. $P(\mathbf{x}'|\mathbf{x},\mathbf{a})$ for $\mathbf{x},\mathbf{a}$ using MLE
    - recompute policy

Visit enough times to estimate $P(\mathbf{x}'|\mathbf{x},\mathbf{a})$?

- How many times are enough?
  - use Chernoff Bound!
- **Chernoff Bound**:
  - $X_1,\ldots,X_n$ are i.i.d. Bernoulli trials with prob. θ
  - $P(|1/n \sum_i X_i - \theta| > \varepsilon) \leq \exp\{-2n\varepsilon^2\}$

Putting it all together

- **Theorem**: With prob. at least 1-δ, Rmax will reach a ε-optimal policy in time polynomial in: num. states, num. actions, T, 1/ε, 1/δ
  - Every T steps:
    - achieve near optimal reward (great!), or
    - visit an unknown state-action pair ! num. states and actions is finite, so can't take too long before all states are known

What you need to know about RL…

- Neither supervised, nor unsupervised learning
- Try to learn to act in the world, as we travel states and get rewards
- Model-based & Model-free approaches
- Rmax, a model based approach:
  - Learn model of rewards and transitions
  - Address exploration-exploitation tradeoff
  - Simple algorithm, great in practice

Closing….

What you have learned this quarter

- Learning is function approximation
- Point estimation
- Regression
- LASSO
- Subgradient
- Stochastic gradient descent
- Coordinate descent
- Discriminative v. Generative learning
- Naïve Bayes
- Logistic regression
- Bias-Variance tradeoff
- Decision trees
- Cross validation
- Boosting
- Instance-based learning
- Perceptron
- SVMs
- Kernel trick
- PAC learning
- Bayes nets
  - representation, parameter and structure learning
- K-means
- EM
- Mixtures of Gaussians
- Dimensionality reduction, PCA
- MDPs
- Reinforcement learning

BIG PICTURE

- Improving the performance at some task though experience!!! ☺
  - before you start any learning task, remember the fundamental questions:

What is the learning problem?

From what experience?

What model?

What loss function are you optimizing?

With what optimization algorithm?

Which learning algorithm?

With what guarantees?

How will you evaluate it?

You have done a lot!!!

- And (hopefully) learned a lot!!!
  - Implemented
    - LASSO
    - LR
    - Perceptron
    - Clustering
    - …
  - Answered hard questions and proved many interesting results
  - Completed (I am sure) an amazing ML project
  - And did excellently on the final!
- Now you are ready for one of the most sought-after careers in industry today!!! ☺

Thank You for the Hard Work!!!