Clarke Differential

Subdifferential and Subgradient

Definition: Given $f : \mathbb{R}^d \to \mathbb{R}$, for every x, the subdifferential set is defined as $\partial_s f(x) \triangleq \{s \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, f(x') \geq f(x) + s^\top (x' - x)\}$. The elements in the subdifferential set are subgradients.

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Subdifferential is not enough

Definition: Given $f : \mathbb{R}^d \to \mathbb{R}$, for every x, the subdifferential set is defined as $\partial_s f(x) \triangleq \{s \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, f(x') \geq f(x) + s^\top (x' - x)\}$. The elements in the subdifferential set are subgradients.

Clarke Differential

Definition: Given $f : \mathbb{R}^d \rightarrow \mathbb{R}$, for every x, the Clark differential is defined as
$\partial f(x) \triangleq \text{conv}\left(\{s \in \mathbb{R}^d : \exists \{x_i\}_{i=1}^{\infty} \rightarrow x, \{\nabla f(x_i)\}_{i=1}^{\infty} \rightarrow s\}\right)$.
The elements in the subdifferential set are subgradients.

When does Clarke differential exists

Definition (Locally Lipschitz): $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipchitz if $\forall x \in \mathbb{R}^d$, there exists a neighborhood S of x, such that f is Lipchitz in S.

Positive Homogeneity

Definition: $f : \mathbb{R}^d \to \mathbb{R}$ is positive homogeneous of degree L if $f(\alpha x) = \alpha^L f(x)$ for any $\alpha \geq 0$.

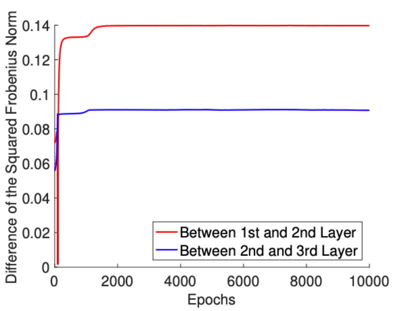
Positive Homogeneity

Positive Homogeneity

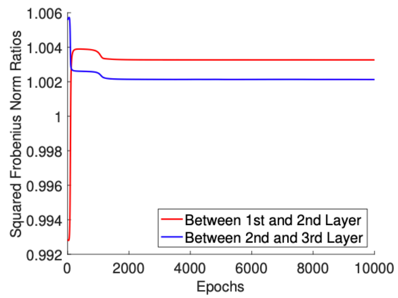
Positive Homogeneity and Clark Differential

Lemma: Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is Locally Lipschitz and L-positively homogeneous. For any $x \in \mathbb{R}^d$ and $s \in \partial f(x)$, we have $\langle s, x \rangle = Lf(x)$.

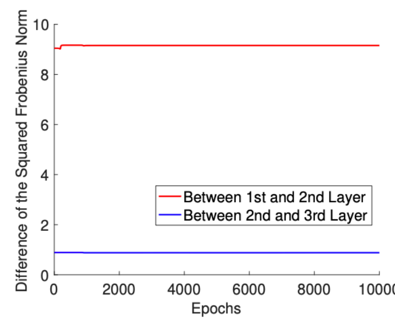
Norm Preservation



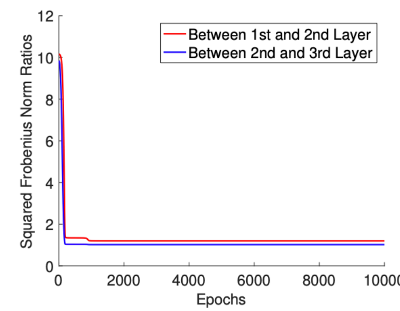
(a) Balanced initialization, squared norm differences.



(b) Balanced initialization, squared norm ratios.



(c) Unbalanced Initialization, squared norm differences.



(d) Unbalanced initialization, squared norm ratios.

Gradient flow and gradient inclusion

Discrete-time dynamics can be complex. Let's use continuous-time dynamics to simplify:

Gradient flow: $x_{t+1} = x_t - \eta \nabla f(x_t) \Rightarrow \frac{x(t)}{dt} = -\nabla f(x(t))$

Gradient inclusion: $\frac{dx(t)}{dt} \in \partial f(x(t))$

Norm preservation by gradient inclusion

Theorem (Du, Hu, Lee '18) Suppose $\alpha > 0$, $f(x; (W_{H+1}, \ldots, \alpha W_i, \ldots, W_1)) = \alpha f(x, (W_{H+1}, \ldots, W_1))$, I.e., predictions are 1-homogeneous in each layer. Then for every pair of layers $(i, j) \in [H+1] \times [H+1]$, the gradient inclusion maintains: for all $t \geq 0$,

$$\frac{1}{2}\|W_h(t)\|_F^2 - \frac{1}{2}\|W_h(0)\|_F^2 = \frac{1}{2}\|W_h(t)\|_F^2 - \frac{1}{2}\|W_{h'}(0)\|_F^2.$$