# Clarke Differential

# Subdifferential and Subgradient

**Definition:** Given $f : \mathbb{R}^d \to \mathbb{R}$, for every $x$, the subdifferential set is defined as $\partial_s f(x) \triangleq \{s \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, f(x') \geq f(x) + s^\top (x' - x)\}$. The elements in the subdifferential set are subgradients.

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# Subdifferential is not enough

**Definition:** Given $f : \mathbb{R}^d \to \mathbb{R}$, for every $x$, the subdifferential set is defined as $\partial_s f(x) \triangleq \{s \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, f(x') \geq f(x) + s^\top (x' - x)\}$. The elements in the subdifferential set are subgradients.

# Clarke Differential

**Definition:** Given $f: \mathbb{R}^d \to \mathbb{R}$, for every $x$, the Clark differential is defined as

$$\partial f(x) \triangleq \operatorname{conv}\left(\{s \in \mathbb{R}^d : \exists \{x_i\}_{i=1}^{\infty} \to x, \{\nabla f(x_i)\}_{i=1}^{\infty} \to s\}\right).$$

The elements in the subdifferential set are subgradients.

# When does Clarke differential exists

**Definition (Locally Lipschitz)**: $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipchitz if $\forall x \in \mathbb{R}^d$, there exists a neighborhood $S$ of $x$, such that $f$ is Lipchitz in $S$.

# Positive Homogeneity

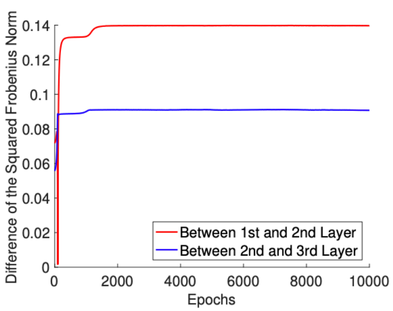
**Definition**: $f : \mathbb{R}^d \to \mathbb{R}$ is positive homogeneous of degree $L$ if $f(\alpha x) = \alpha^L f(x)$ for any $\alpha \geq 0$.

# Positive Homogeneity

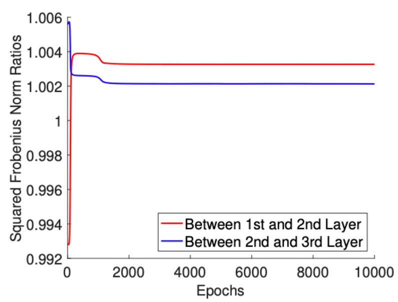
# Positive Homogeneity and Clark Differential

**Lemma:** Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is Locally Lipschitz and $L$ -positively homogeneous. For any $x \in \mathbb{R}^d$ and $s \in \partial f(x)$, we have $\langle s, x \rangle = Lf(x)$.

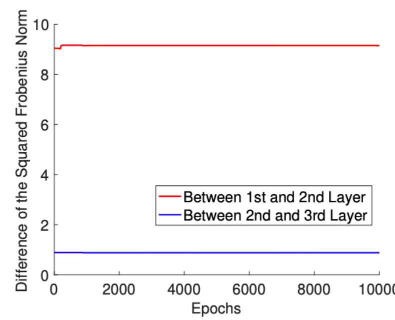
# Norm Preservation



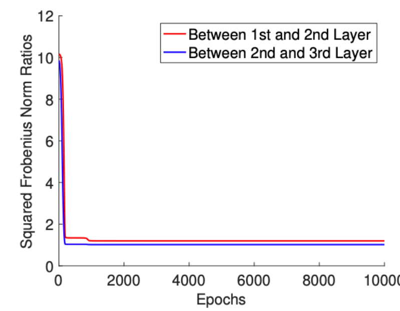

(a) Balanced initialization, squared norm differences.




(b) Balanced initialization, squared norm ratios.




(c) Unbalanced Initialization, squared norm differences.




(d) Unbalanced initialization, squared norm ratios.

# Gradient flow and gradient inclusion

Discrete-time dynamics can be complex. Let's use continuous-time dynamics to simplify:

Gradient flow: $x_{t+1} = x_t - \eta \nabla f(x_t) \Rightarrow \frac{x(t)}{dt} = -\nabla f(x(t))$

Gradient inclusion: $\frac{dx(t)}{dt} \in \partial f(x(t))$

# Norm preservation by gradient inclusion

**Theorem** (Du, Hu, Lee '18) Suppose $\alpha > 0$, $f(x; (W_{H+1}, \ldots, \alpha W_i, \ldots, W_1)) = \alpha f(x, (W_{H+1}, \ldots, W_1))$, I.e., predictions are 1-homogeneous in each layer. Then for every pair of layers $(i, j) \in [H+1] \times [H+1]$, the gradient inclusion maintains: for all $t \geq 0$,

$$\frac{1}{2}\|W_h(t)\|_F^2 - \frac{1}{2}\|W_h(0)\|_F^2 = \frac{1}{2}\|W_h(t)\|_F^2 - \frac{1}{2}\|W_{h'}(0)\|_F^2.$$

# Norm preservation by gradient inclusion

# Optimization Methods for Deep Learning

# Gradient descent for non-convex optimization

**Decsent Lemma:** Let $f:\mathbb{R}^d \to \mathbb{R}$ be twice differentiable, and $\|\nabla^2 f\|_2 \le \beta$. Then setting the learning rate $\eta = 1/\beta$, and applying gradient descent, $x_{t+1} = x_t - \eta \nabla f(x_t)$, we have:

$$f(x_t) - f(x_{t+1}) \ge \frac{1}{2\beta} \|\nabla f(x_t)\|_2^2.$$

# Converging to stationary points

**Theorem:** In $T = O(\frac{\beta}{\epsilon^2})$ iterations, we have $\|\nabla f(x)\|_2 \leq \epsilon$.

# Gradient Descent for Quadratic Functions

**Problem:** $\min_x \frac{1}{2} x^\top A x$ with $A \in \mathbb{R}^{d\times d}$ being positive-definite.

**Theorem:** Let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and the smallest eigenvalues of $A$. If we set $\eta \leq \frac{1}{\lambda_{\max}}$, we have

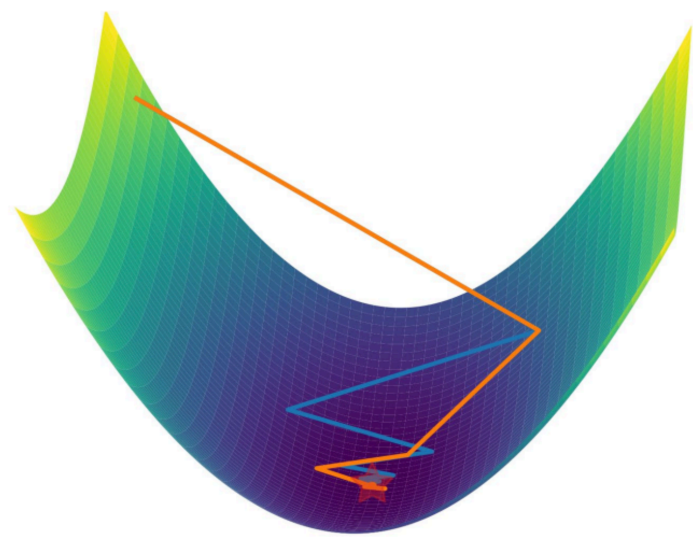
$$\|x_t\|_2 \leq \left(1 - \eta\lambda_{\min}\right)^t \|x_0\|_2$$

# Momentum: Heavy-Ball Method (Polyak '64)

**Problem:** $\min_x f(x)$

**Method:** $v_{t+1} = -\nabla f(x_t) + \beta v_t$

$$x_{t+1} = x_t + \eta v_{t+1}$$

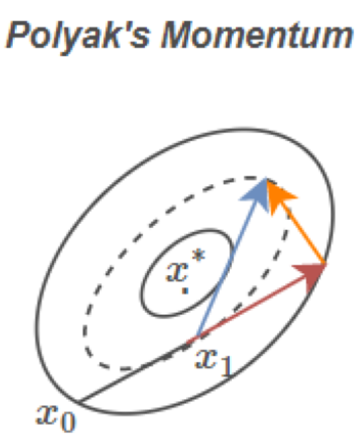


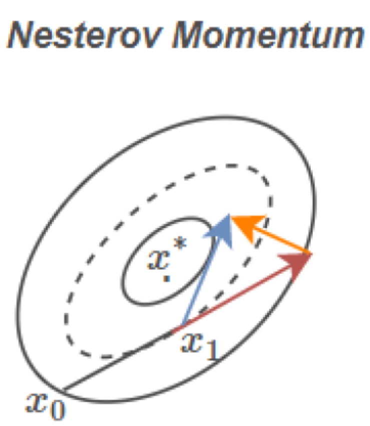
# Momentum: Nesterov Acceleration (Nesterov '89)

**Problem:** $\min_x f(x)$

**Method:** $v_{t+1} = -\nabla f(x_t + \beta v_t) + \beta v_t$

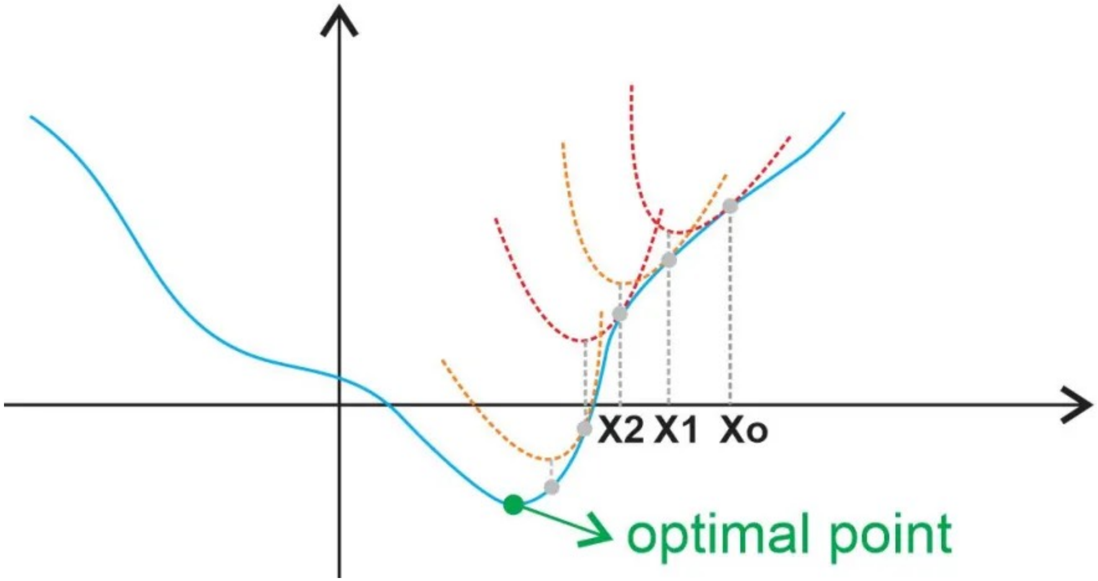
$$x_{t+1} = x_t + \eta v_{t+1}$$



# Newton's Method

**Newton's Method:** $x_{t+1} = x_t - \eta(\nabla^2 f(x_t))^{-1} \nabla f(x_t)$



# AdaGrad (Duchi et al. '11)

**Newton Method:** $x_{t+1} = x_t - \eta(\nabla^2 f(x_t))^{-1} \nabla f(x_t)$

**AdaGrad:** separate learning rate for every parameter

$$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1} \nabla f(x_t),\ (G_t)_{ii} = \sqrt{\sum_{j=1}^{t-1} \left( \nabla f(x_t)_i \right)^2}$$

# RMSProp (Hinton et al. '12)

**AdaGrad:** separate learning rate for every parameter

$$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1} \nabla f(x_t),\ (G_t)_{ii} = \sqrt{\sum_{j=1}^{t-1} \left( \nabla f(x_t)_i \right)^2}$$

**RMSProp:** exponential weighting of gradient norms

$$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1/2} \nabla f(x_t),$$

$$(G_{t+1})_{ii} = \beta(G_t)_{ii} + (1-\beta)(\nabla f(x_t)_i)^2$$

# AdaDelta (Zeiler '12)

**RMSProp:**

$$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1/2} \nabla f(x_t),$$
$$(G_{t+1})_{ii} = \beta(G_t)_{ii} + (1-\beta)(\nabla f(x_t)_i)^2$$

**AdaDelta:**

$$x_{t+1} = x_t - \eta \Delta x_t,$$
$$\Delta x_t = \sqrt{u_t + \epsilon} \cdot (G_{t+1} + \epsilon I)^{-1/2} \nabla f(x_t)$$
$$(G_{t+1})_{ii} = \rho(G_t)_{ii} + (1-\rho)(\nabla f(x_t)_i)^2,$$
$$u_{t+1} = \rho u_t + (1-\rho)\|\Delta x_t\|_2^2$$

# Adam (Kingma & Ba '14)

**Momentum:**

$v_{t+1} = -\nabla f(x_t) + \beta v_t, x_{t+1} = x_t + \eta v_{t+1}$

**RMSProp:** exponential weighting of gradient norms

$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1} \nabla f(x_t),$
$(G_t)_{ii} = \beta(G_t)_{ii} + (1-\beta)(\nabla f(x_t)_i)^2$

**Adam**

$v_{t+1} = \beta_1 v_t + (1-\beta_1) \nabla f(x_t)$
$(G_{t+1})_{ii} = \beta_2(G_t)_{ii} + (1-\beta_2)(\nabla f(x_t)_i)^2$
$x_{t+1} = x_t - \eta(G_{t+1} + \epsilon I)^{-1/2} v_{t+1}$

Default choice nowadays.

# Other Optimizers

- AdamW
- NAdam
- RAdam
- GGT
- K-FAC
- …