

The (Infinite-Horizon Discounted) MDP

A **Markov Decision Process** is a tuple $M = (\mathcal{S}, \mathcal{A}, P, r, \gamma, \mu)$:

- \mathcal{S} : state space (assume finite)
- \mathcal{A} : action space (assume finite)
- $P : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$: transition kernel
 $P(s' | s, a)$: prob. of moving to s'
- $r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$: reward function
- $\gamma \in [0, 1)$: discount factor
- $\mu \in \Delta(\mathcal{S})$: initial state distribution

Goal: Maximize $\mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s \right]$

Interaction Protocol

$s_0 \sim \mu$. At each step $t = 0, 1, 2, \dots$:

- 1 Observe s_t
- 2 Select $a_t \in \mathcal{A}$
- 3 Receive $r_t = r(s_t, a_t)$
- 4 Transition $s_{t+1} \sim P(\cdot | s_t, a_t)$

Examples

Navigation

State = current location. Actions = {N, S, E, W}. Deterministic transitions.
Reward = 1 at goal, else 0. Discount γ incentivizes shortest paths.
Optimal policy: greedy shortest path. Value $\approx \gamma^d$ for path length d .

Inventory Management

State = current stock level. Action = how much inventory to order each day.
Demand is stochastic: reward = sales revenue (only if item is in stock) – ordering cost.
Tension: understocking loses sales; overstocking ties up capital.

Strategic Games

State = board position. Actions = legal moves.
Reward = eventual win/loss.
RL has achieved superhuman performance in Go, Chess, Poker, Backgammon.

Planning vs. Statistical RL

Planning (dynamics known)

The model P, r is given. The challenge is purely *computational*: how do we efficiently find the optimal policy?

Examples:

- Shortest path / navigation
- Board games (Chess, Go): simulator provides exact next state
- Video games: emulator is the model

Statistical RL (dynamics unknown)

We don't know P or r . We must *learn* what actions lead to what outcomes by interacting with the environment.

Examples:

- Personalized medical treatment: how does this patient's condition evolve if we prescribe drug A vs. drug B ?
- Inventory management: what is the true demand distribution?

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Focus of this course

We study the **statistical** problem: how many interactions does it take to learn a near-optimal policy? Planning serves as a subroutine, but the core challenge is exploration and generalization.

Policies and Value Functions

History: $\tau_t = (s_0, a_0, r_0, s_1, \dots, s_t)$.

A **policy** π maps histories to distributions over actions.

Stationary Policy

$\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$ — actions depend only on current state s_t .

A **deterministic** stationary policy is $\pi : \mathcal{S} \rightarrow \mathcal{A}$.

Value function $V^\pi : \mathcal{S} \rightarrow \mathbb{R}$:

$$V^\pi(s) = \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid \pi, s_0 = s \right], \quad 0 \leq V^\pi(s) \leq \frac{1}{1-\gamma}.$$

Action-value (Q-value) function $Q^\pi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$:

$$Q^\pi(s, a) = \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid \pi, s_0 = s, a_0 = a \right].$$

Goal: find π maximizing $V^\pi(s)$ for all s (or for $s \sim \mu$).

Bellman Consistency Equations

Lemma 1.4 (Bellman consistency for π)

For any stationary policy π , for all $s \in \mathcal{S}$ and $a \in \mathcal{A}$:

$$V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[Q^\pi(s, a)]$$
$$Q^\pi(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)}[V^\pi(s')]$$

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Proof. Unroll one step using the Markov property and tower property:

$$\begin{aligned}Q^\pi(s, a) &= \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid \pi, s_0 = s, a_0 = a \right] \\&= r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[\mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(s_{t+1}, a_{t+1}) \mid \pi, s_1 = s' \right] \right] \\&= r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)}[V^\pi(s')].\end{aligned}$$

The first equation in the lemma then follows by averaging over $a \sim \pi(\cdot|s)$. □

Matrix Form & Corollary 1.5

In matrix/vector form (treating Q^π, r as vectors over $\mathcal{S} \times \mathcal{A}$):

$$Q^\pi = r + \gamma P V^\pi = r + \gamma P^\pi Q^\pi,$$

where $P_{(s,a),(s',a')}^\pi = P(s'|s, a) \pi(a'|s')$ is the induced state-action transition matrix.

Corollary 1.5. (Closed-form solution)

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$$Q^\pi = (I - \gamma P^\pi)^{-1} r.$$

Proof. It suffices to show $I - \gamma P^\pi$ is invertible. For any nonzero $x \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$:

$$\begin{aligned} \|(I - \gamma P^\pi)x\|_\infty &\geq \|x\|_\infty - \gamma \|P^\pi x\|_\infty && \text{(triangle inequality)} \\ &\geq \|x\|_\infty - \gamma \|x\|_\infty && (P^\pi \text{ row-stochastic}) \\ &= (1 - \gamma) \|x\|_\infty > 0. && (\gamma < 1, x \neq 0) \end{aligned}$$

So $I - \gamma P^\pi$ has trivial kernel, hence is invertible. Then $Q^\pi = r + \gamma P^\pi Q^\pi$ gives $(I - \gamma P^\pi)Q^\pi = r$. □

Optimal Policy: The Key Theorem

Theorem 1.7 (Existence of optimal stationary policy)

Define $V^*(s) \triangleq \sup_{\pi} V^{\pi}(s)$ and $Q^*(s, a) \triangleq \sup_{\pi} Q^{\pi}(s, a)$.

There exists a stationary deterministic policy π^* such that for all s, a :

$$V^{\pi^*}(s) = V^*(s), \quad Q^{\pi^*}(s, a) = Q^*(s, a).$$

Why is this surprising? The supremum is over *all* policies (non-stationary, randomized, history-dependent). Yet a simple “look only at current state” policy suffices.

Theorem 1.7: Proof sketch

Step 1. $\sup_{\pi} \mathbb{E}[\sum_{t=1}^{\infty} \gamma^t r(S_t, A_t) \mid \pi, S_1 = s'] = \gamma V^*(s').$ (*shift time index*)

Step 2. $\tilde{\pi}(s) \in \arg \max_a \mathbb{E}_{\tilde{\pi}}[r(s, a) + \gamma V^*(S_1) \mid s].$

Step 3. Trivially we have $V^{\tilde{\pi}}(s_0) \leq V^*(s_0).$ Now show $V^*(s_0) \leq V^{\tilde{\pi}}(s_0):$

$$\begin{aligned} V^*(s_0) &= \sup_{\pi} \mathbb{E}_{\pi} \left[r(S_0, A_0) + \sum_{t=1}^{\infty} \gamma^t r(S_t, A_t) \mid s_0 \right] \\ &= \sup_{\pi} \mathbb{E}_{\pi} \left[r(S_0, A_0) + \mathbb{E}_{\pi} \left[\sum_{t=1}^{\infty} \gamma^t r(S_t, A_t) \mid \pi, s_0, A_0, R_0, S_1 \right] \mid s_0 \right] \\ &\leq \sup_{\pi} \mathbb{E}_{\pi} \left[r(S_0, A_0) + \sup_{\pi'} \mathbb{E}_{\pi'} \left[\sum_{t=1}^{\infty} \gamma^t r(S_t, A_t) \mid \pi', s_0, A_0, R_0, S_1 \right] \mid s_0 \right] \\ &= \sup_{\pi} \mathbb{E}_{\pi} \left[r(S_0, A_0) + \gamma V^*(S_1) \mid s_0 \right] && \text{(Step 1)} \\ &= \mathbb{E}_{\tilde{\pi}} \left[r(S_0, A_0) + \gamma V^*(S_1) \mid s_0 \right] && \text{(def. } \tilde{\pi} \text{)} \\ &\leq \mathbb{E}_{\tilde{\pi}} \left[r(S_0, A_0) + \gamma r(S_1, A_1) + \gamma^2 V^*(S_2) \mid s_0 \right] && \text{(recurse)} \\ &\leq V^{\tilde{\pi}}(s_0) \end{aligned}$$

Bellman Optimality Equations

Define the **Bellman optimality operator** $T_M : \mathbb{R}^{|S||A|} \rightarrow \mathbb{R}^{|S||A|}$:

$$(TQ)(s, a) \triangleq r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \left[\max_{a'} Q(s', a') \right].$$

Write $V_Q(s) \triangleq \max_a Q(s, a)$ and $\pi_Q(s) \in \arg \max_a Q(s, a)$ (greedy policy).

Theorem 1.8 (Bellman optimality equations)

$Q = Q^*$ if and only if $Q = TQ$.

Moreover, π_{Q^*} is an optimal policy.

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$Q = Q^*$ if and only if $Q = TQ$.

Moreover, π_{Q^*} is an optimal policy.

This almost gives us an algorithm for identifying π^* : just find some Q that satisfies the fixed point equation $Q = TQ$!

Q-Value Iteration

Algorithm: Initialize $Q^{(0)} = 0$. Repeat:

$$Q^{(k+1)} \leftarrow TQ^{(k)}, \quad \text{i.e.,} \quad Q^{(k+1)}(s, a) = r(s, a) + \gamma \mathbb{E}_{s'} \left[\max_{a'} Q^{(k)}(s', a') \right].$$

Extract policy: $\pi^{(k)} = \pi_{Q^{(k)}}$ (greedy w.r.t. $Q^{(k)}$).

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Why does this converge? The operator T is a γ -**contraction** in $\|\cdot\|_\infty$, and Q^* is its unique fixed point.

Lemma 1.10 (Contraction)

For any $Q, Q' \in \mathbb{R}^{|S||A|}$:

$$\|TQ - TQ'\|_\infty \leq \gamma \|Q - Q'\|_\infty.$$

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Recall $TQ^* = Q^*$ and $Q_0 = 0$. Thus,

$$\|Q_k - Q^*\|_\infty = \|TQ_{k-1} - TQ^*\|_\infty \leq \gamma \|Q_{k-1} - Q^*\|_\infty \leq \gamma^k \|Q^*\|_\infty \leq \frac{\gamma^k}{1-\gamma}.$$

Lemma 1.10: Contraction — Proof

Step 1. Fix any s , WLOG $V_Q(s) \geq V_{Q'}(s)$. Let $a^* \in \arg \max_a Q(s, a)$.

$$V_Q(s) - V_{Q'}(s) = Q(s, a^*) - \max_a Q'(s, a) \leq Q(s, a^*) - Q'(s, a^*) \leq \|Q - Q'\|_\infty.$$

Thus, $\|V_Q - V_{Q'}\|_\infty \leq \|Q - Q'\|_\infty$.

Step 2.

$$\|TQ - TQ'\|_\infty = \gamma \|P(V_Q - V_{Q'})\|_\infty \leq \gamma \|V_Q - V_{Q'}\|_\infty \leq \gamma \|Q - Q'\|_\infty,$$

where $\|Px\|_\infty \leq \|x\|_\infty$ because each row of P is a probability distribution. \square

Q-Error Amplification (Lemma 1.11)

Lemma 1.11 (Suboptimality of greedy policy)

For any $Q \in \mathbb{R}^{|S||A|}$:

$$V^{\pi_Q} \geq V^* - \frac{2 \|Q - Q^*\|_\infty}{1 - \gamma} \cdot \mathbf{1}.$$

We showed above that $\|Q_k - Q^*\|_\infty \leq \frac{\gamma^k}{1 - \gamma}$. Thus, $V^{\pi_{Q_k}} \geq V^* - \frac{2\gamma^k}{(1 - \gamma)^2} \cdot \mathbf{1}$.

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Theorem 1.12 (Q-value iteration convergence)

Set $Q^{(0)} = 0$. After $k \geq \frac{1}{1 - \gamma} \log \frac{2}{(1 - \gamma)^2 \varepsilon}$ iterations,

$$V^{\pi^{(k)}} \geq V^* - \varepsilon \cdot \mathbf{1}.$$

Lemma 1.11: Proof

Fix s ; let $a = \pi_Q(s)$ and π^* be optimal. Then:

$$V^*(s) - V^{\pi_Q}(s) = \underbrace{Q^*(s, \pi^*(s)) - Q^*(s, a)}_{(I)} + \underbrace{Q^*(s, a) - Q^{\pi_Q}(s, a)}_{(II)}.$$

Bounding (I): Since π_Q is greedy w.r.t. Q , we have $Q(s, \pi^*(s)) \leq Q(s, a)$, so:

$$(I) = Q^*(s, \pi^*(s)) - Q^*(s, a) \leq [Q^*(s, \pi^*) - Q(s, \pi^*)] + [Q(s, a) - Q^*(s, a)] \leq 2 \|Q - Q^*\|_\infty.$$

Bounding (II): By Bellman consistency for π_Q :

$$(II) = \gamma \mathbb{E}_{s'} [V^*(s') - V^{\pi_Q}(s')] \leq \gamma \|V^* - V^{\pi_Q}\|_\infty.$$

Combine: $\|V^* - V^{\pi_Q}\|_\infty \leq 2 \|Q - Q^*\|_\infty + \gamma \|V^* - V^{\pi_Q}\|_\infty.$

Rearranging:

$$\|V^* - V^{\pi_Q}\|_\infty \leq \frac{2}{1 - \gamma} \|Q - Q^*\|_\infty. \quad \square$$

Policy Iteration

Start from any deterministic stationary policy π_0 . Repeat for $k = 0, 1, 2, \dots$:

Step 1: Policy Evaluation

Solve for Q^{π_k} : $Q^{\pi_k} = (I - \gamma P^{\pi_k})^{-1} r$.

Step 2: Policy Improvement

$\pi_{k+1} = \pi_{Q^{\pi_k}}$ (greedy w.r.t. Q^{π_k}).

Key properties:

- Values improve monotonically: $Q^{\pi_{k+1}} \geq Q^{\pi_k}$ (elementwise).
- Converges geometrically to Q^* .
- Can terminate with *exact* π^* in finitely many steps.

Policy Iteration: Monotonicity & Contraction

Lemma 1.13

- 1 **(Sandwich)** $Q^{\pi_{k+1}} \geq TQ^{\pi_k} \geq Q^{\pi_k}$.
- 2 **(Geometric contraction)** $\|Q^* - Q^{\pi_{k+1}}\|_{\infty} \leq \gamma \|Q^* - Q^{\pi_k}\|_{\infty}$.

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(1a) $TQ^{\pi_k} \geq Q^{\pi_k}$: Since $V_{Q^{\pi_k}}(s) = \max_a Q^{\pi_k}(s, a) \geq Q^{\pi_k}(s, \pi_k(s)) = V^{\pi_k}(s)$:

$$TQ^{\pi_k}(s, a) = r(s, a) + \gamma \mathbb{E}_{s'}[V_{Q^{\pi_k}}(s')] \geq r(s, a) + \gamma \mathbb{E}_{s'}[V^{\pi_k}(s')] = Q^{\pi_k}(s, a).$$

(1b) $Q^{\pi_{k+1}} \geq TQ^{\pi_k}$: Let $\pi' = \pi_{k+1}$. The policy evaluation operator

$T^{\pi'} V(s) = r(s, \pi'(s)) + \gamma \mathbb{E}_{s'}[V(s')]$ satisfies:

$$T^{\pi'} V^{\pi_k}(s) = Q^{\pi_k}(s, \pi'(s)) = V_{Q^{\pi_k}}(s) \geq V^{\pi_k}(s).$$

Iterating monotonicity: $V^{\pi_{k+1}} = \lim_{n \rightarrow \infty} (T^{\pi'})^n V^{\pi_k} \geq V_{Q^{\pi_k}}$. Therefore:

$$Q^{\pi_{k+1}}(s, a) = r(s, a) + \gamma \mathbb{E}_{s'}[V^{\pi_{k+1}}(s')] \geq r(s, a) + \gamma \mathbb{E}_{s'}[V_{Q^{\pi_k}}(s')] = TQ^{\pi_k}(s, a).$$

Policy Iteration: Monotonicity & Contraction

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① **(Sandwich)** $Q^{\pi_{k+1}} \geq TQ^{\pi_k} \geq Q^{\pi_k}$.

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(1b) $Q^{\pi_{k+1}} \geq TQ^{\pi_k}$: Let $\pi' = \pi_{k+1}$. The policy evaluation operator $T^{\pi'} V(s) = r(s, \pi'(s)) + \gamma \mathbb{E}_{s'}[V(s')]$ satisfies:

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(2) Contraction: Since $Q^* \geq Q^{\pi_{k+1}} \geq TQ^{\pi_k}$:

$$0 \leq Q^* - Q^{\pi_{k+1}} \leq Q^* - TQ^{\pi_k} = TQ^* - TQ^{\pi_k}.$$

Taking $\|\cdot\|_\infty$ and applying Lemma 1.10: $\|Q^* - Q^{\pi_{k+1}}\|_\infty \leq \gamma \|Q^* - Q^{\pi_k}\|_\infty$

Policy Iteration: Convergence

Theorem 1.14 (Policy iteration convergence)

Starting from any π_0 :

$$\|Q^* - Q^{\pi_k}\|_{\infty} \leq \gamma^k \|Q^* - Q^{\pi_0}\|_{\infty} \leq \frac{\gamma^k}{1 - \gamma}.$$

For $k \geq \frac{\log(1/((1 - \gamma)\epsilon))}{1 - \gamma}$: $Q^{\pi_k} \geq Q^* - \epsilon \cdot \mathbf{1}$.

Policy iteration vs. value iteration:

- Same geometric rate γ^k , but PI terminates with *exact* π^* (finitely many policies).
- Naive bound: $|\mathcal{A}|^{|\mathcal{S}|}$ iterations.
- Each PI iteration costs $O(|\mathcal{S}|^3 + |\mathcal{S}|^2|\mathcal{A}|)$ (policy evaluation + greedy step).

Finite-Horizon MDPs

A **finite-horizon MDP** $M = (\mathcal{S}, \mathcal{A}, \{P_h\}_{h=0}^{H-1}, \{r_h\}_{h=0}^{H-1}, H, \mu)$:

- Horizon $H \in \mathbb{N}$; rewards $r_h(s, a) \in [0, 1]$; transitions P_h may depend on h .

Value functions: For $h \in \{0, \dots, H\}$, $s \in \mathcal{S}$:

$$V_h^\pi(s) = \mathbb{E} \left[\sum_{t=h}^{H-1} r_t(S_t, A_t) \mid \pi, S_h = s \right], \quad V_H^\pi(s) \equiv 0.$$

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Stage- h Bellman optimality operator:

$$(T_h f)(s, a) = r_h(s, a) + \mathbb{E}_{s' \sim P_h(\cdot | s, a)} \left[\max_{a'} f(s', a') \right].$$

Theorem 1.9 (Bellman optimality: finite horizon)

The unique solution to $Q_H^* \equiv 0$, $Q_h^* = T_h Q_{h+1}^*$ equals $\sup_\pi Q_h^\pi$.
The policy $\pi^*(s, h) \in \arg \max_a Q_h^*(s, a)$ is optimal.

Proof by backward induction. (Exercise.)

Discounted vs. Finite Horizon

Infinite-Horizon Discounted

- Stationary dynamics/rewards
- Tabular size: $O(|\mathcal{S}||\mathcal{A}|)$
- Convergence via contraction ($\gamma < 1$)
- Effective horizon: $\frac{1}{1-\gamma}$

Finite Horizon (Time-Dependent)

- P_h, r_h may vary with h
- Tabular size: $O(H|\mathcal{S}||\mathcal{A}|)$
- Backward induction (exact, no contraction needed)
- Sharper theoretical rates

Convention in this course

Both settings studied. The finite-horizon time-dependent formulation is often **more convenient theoretically** (backward induction, sharper rates).

Can fold time h into the state to recover a stationary MDP on $\mathcal{S} \times [H]$.

Backward Induction (Section 1.3.3)

In the finite-horizon setting, backward induction computes Q_h^* **exactly** in $O(H|\mathcal{S}|^2|\mathcal{A}|)$ time.

Algorithm

Initialize $Q_H(s, a) \equiv 0$. For $h = H - 1, H - 2, \dots, 0$:

$$Q_h(s, a) \leftarrow r_h(s, a) + \mathbb{E}_{s' \sim P_h(\cdot | s, a)} \left[\max_{a'} Q_{h+1}(s', a') \right].$$

By Theorem 1.9, the resulting $Q_h = Q_h^*$. The optimal policy is $\pi^*(s, h) = \arg \max_a Q_h^*(s, a)$.

No contraction needed: backward induction is exact. This is a major advantage of the finite-horizon formulation.

Complexity: $O(H|\mathcal{S}|^2|\mathcal{A}|)$ with naive matrix-vector products, $O(H|\mathcal{S}||\mathcal{A}|)$ with sparse transitions.

Summary

MDP Basics:

- MDP tuple $(\mathcal{S}, \mathcal{A}, P, r, \gamma, \mu)$
- Policies, value functions V^π, Q^π
- Bellman consistency (Lemma 1.4)
- Optimal policy exists and is stationary (Thm. 1.7)
- Bellman optimality equations (Thm. 1.8)
- Finite-horizon formulation (Thm. 1.9)

Planning:

- Value iteration: T is γ -contraction (Lem. 1.10), convergence (Thm. 1.12)
- Suboptimality of greedy policy (Lem. 1.11): $\frac{2}{1-\gamma} \|Q - Q^*\|_\infty$
- Policy iteration: monotone + geometric convergence (Lem. 1.13, Thm. 1.14)
- Backward induction: exact, $O(H|\mathcal{S}|^2|\mathcal{A}|)$

Next

Offline RL: given a fixed dataset of transitions, how do we evaluate or optimize a policy?