Toss two coins, come up H or T

- Sample space $\Omega = \{TT, TH, HT, HH\}$
- Outcome $\omega \in \Omega$ $\quad \omega = \{TT\}$
- σ-algebra called $\mathcal{F}$ is a collection of subsets of Ω s.t.
  1. $\Omega \in \mathcal{F}$
  2. If $F \in \mathcal{F}$ then $F^c \in \mathcal{F}$
  3. For a sequence F_n, $\bigcup_n F_n \in \mathcal{F}$

$$\mathcal{F} = \{\emptyset, \Omega, TT, TH, HT, HH, \{TT, TH\}, \{TT, HT\}, \{\ \}, \{TH, HT, HH\}, \ldots \{\ \}\}.$$

- Probability measure for a σ-algebra $\mathcal{F}$ on Ω satisfies
  1. $\mathbb{P} : \mathcal{F} \to [0,1]$
  2. $\mathbb{P}(\emptyset) = 0$
  3. $\mathbb{P}(\Omega) = 1$
  4. For a seq $F_n \in \mathcal{F}$ w/ $F_i \cap F_j = \emptyset$, $\mathbb{P}(\bigcup_n F_n) = \sum_n \mathbb{P}(F_n)$
- A probability space is defined as tuple $(\Omega, \mathcal{F}, \mathbb{P})$
- Fix $(\Omega, \mathcal{F}, \mathbb{P})$. A function $g : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if for all Borel sets B we have $\{\omega : g(\omega) \in B\} \in \mathcal{F}$.
- Random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$ that is $\mathcal{F}$-measurable.
- Let X_t be a set of RV.s indexed by $t \in T$. on space

$(\Omega, \mathcal{F}, P)$, we define $\sigma(\{X_t\}_{t\in T})$ as the natural σ-algebra for $\{X_t\}_{t\in T}$ to be the smallest σ-algebra for which $\{X_t\}_{t\in T}$ are measurable.

X be the R.V. counting # of heads

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \in TT \\ 1 & \text{if } \omega \in \{TH, HT\} \\ 2 & \text{if } \omega \in \{HH\} \end{cases}$$

$\{\omega : X(\omega) = 0\} = \{TT\}$

$\{\omega : X(\omega) = 1\} = \{TH\}, \{H, T\}$

$\{\omega : X(\omega) = 2\} = \{HH\}$

$\{\omega : X(\omega) \notin \{0, 1, 2\}\} = \emptyset$

$\sigma(X) = \emptyset, TT, \{HT, TH\}, \{HH\}, \{TT, HT, TH\}$
$\{TT, HH\}, \{HT, TH, HH\}, \{TT, HT, TH, HH\}$

$$Y(\omega) = \begin{cases} 0 & \text{if } \omega = TT \\ 1 & \text{if } \omega \in \{HT, TH, HH \end{cases}$$

$\sigma(Y) = \emptyset, \{TT\}, \{HT, TH, HH\}, \Omega$

$\sigma(Y) \subset \sigma(X) \subset \mathcal{F}$

Assume X, Y are discrete

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(\omega = HT \mid X=1) = \frac{\mathbb{P}(\omega = HT,\ X=1)}{\mathbb{P}(X=1)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\mathbb{E}[X \mid A] := \sum_x x\, \mathbb{P}(X = x \mid A)$$

(ex. $A = \{Y=1\}$)

In continuous R.V. $\exists$ pathological space

s.t. $\nexists g$ s.t. $g(Y(\omega)) = \mathbb{E}[X \mid Y](\omega)$

Conditional Expectation. Let X be a R.V. on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{H} \subseteq \mathcal{F}$. Then we say $\mathbb{E}[X \mid \mathcal{H}] : \mathcal{H} \to \mathbb{R}$ is the cond. expectation of X given $\mathcal{H}$ if

$$\int_{\omega \in H} \mathbb{E}[X \mid \mathcal{H}](\omega)\, d\mathbb{P}(\omega) = \int_{\omega \in H} X(\omega)\, d\mathbb{P}(\omega). \quad \forall H \in \mathcal{H}$$

$\mathbb{E}[X \mid \sigma(X)] = X$

$\mathbb{E}[X \mid \sigma(Y)] = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$

Filtrations

Let $X_1, X_2, X_3, \ldots$ be a sequence of R.V. defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{F}_t = \sigma(\{X_s\}_{s \le t})$

We say $\mathbb{F} = \{\mathcal{F}_t\}_t$ is a filtration on $\mathcal{F}$ if $\{\Omega, \emptyset\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$

We say $\{X_t\}_{t=1}^n$ is $\mathbb{F}$ adapted if X_t is $\mathcal{F}_t$ measurable.

We call $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space.

Martingales

An $\mathbb{F}$-adapted sequence of RV. $\{X_t\}_t$ is an $\mathbb{F}$-adapted martingale if $\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = X_t$ and $\mathbb{E}[|X_t|] < \infty$ $\forall t$.

Ex. $Z_t \sim N(0,1)$, $X_t = \sum_{s \le t} Z_s$ then X_t martingale

$$
\begin{aligned}
\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}[Z_{t+1} + X_t \mid \mathcal{F}_t] \\
&= \mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] + \mathbb{E}[X_t \mid \mathcal{F}_t] \\
&= \mathbb{E}[Z_{t+1}] + X_t \\
&= X_t
\end{aligned}
$$

- We say X_t is a supermartingale if $\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] \leq X_t$
- We say X_t is a submartingale if $\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] \geq X_t$

$$
X_t = \exp\left(\lambda \sum_{s=1}^{t} Z_s - \lambda^2 t/2\right), \quad Z_s \sim \mathcal{N}(0,1)
$$

$$
\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = \mathbb{E}\left[\exp(\lambda Z_{t+1} - \lambda^2/2)\exp\left(\lambda \sum_{s=1}^{t} Z_s - \lambda^2 t/2\right) \Big| \mathcal{F}_t\right]
$$

$$
= \underbrace{\mathbb{E}\left[\exp(\lambda Z_{t+1} - \lambda^2/2) \mid \mathcal{F}_t\right]}_{\leq 1} \cdot \underbrace{\mathbb{E}\Big[\underbrace{\exp\left(\lambda \sum_{s=1}^{t} Z_s - \lambda^2 t/2\right)}_{= X_t} \Big| \mathcal{F}_t\Big]}_{X_t}
$$

$$
\leq X_t
$$

A random variable $\tau \in \mathbb{N}$ is a <u>stopping time</u> wrt $\mathbb{F}$ if $\mathbb{1}\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable. $\forall t$

$X_t = \sum_{s=1}^{t} Z_s$, $Z_s \sim \mathcal{N}(0,1)$, $\tau = \min\{t : X_t > \varepsilon\}$

valid stopping time

$\tau' = \max\{t : X_t > \varepsilon\}$

Doob's Optional Stopping Theorem. Let $\mathbb{F}$ be a filtration of $\{X_t\}_t$ martingale sequence and let τ be an $\mathbb{F}$-adapted stopping time. If

- $\exists N \in \mathbb{N} : \mathbb{P}(\tau < N) = 1$, or
- $\mathbb{E}[\tau] < \infty$ and $\mathbb{E}[|X_{t+1} - X_t|] < c \quad \forall t$

then X_τ is well-defined and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Furthermore, if X_t is supermartingale then $\mathbb{E}[X_\tau] \le \mathbb{E}[X_0]$

" " submartingale then $\mathbb{E}[X_\tau] \ge \mathbb{E}[X_0]$

Ex. Wald's identity. Let $Z_t = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$ and

$X_t = \sum_{s=1}^{t} Z_s$. $\tau = \min\{t : X_t \ge \varepsilon\}$. assume $\varepsilon \in \mathbb{N}$

$$M_t = X_t - tp$$

$$\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] = \mathbb{E}[(Z_{t+1} - p) + X_t - tp \mid \mathcal{F}_t]$$
$$= X_t - tp = M_t$$

$$\mathbb{E}[X_\tau - p\tau] = \mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0$$

$$\Rightarrow \varepsilon = \mathbb{E}[X_\tau] = p \cdot \mathbb{E}[\tau]$$

$$\Rightarrow \quad \mathbb{E}[\tau] = \frac{\varepsilon}{p}$$

<u>Maximal inequality</u> Let X_t be an $\mathbb{F}$-adapted seq. of R.V. w/ $X_t \geq 0$. Then for any $\varepsilon > 0$

- If X_t is a supermartingale $\mathbb{P}\left(\max_{t \in \mathbb{N}} X_t > \varepsilon\right) \leq \frac{\mathbb{E}[X_0]}{\varepsilon}$.
- If X_t is a submartingale $\mathbb{P}\left(\max_{t=1,\dots,n} X_t > \varepsilon\right) \leq \frac{\mathbb{E}[X_n]}{\varepsilon}$.