Universal Portfolio Optimization

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Universal Portfolio Optimization

Given a collection of stocks, let the ith stock have price $S_t(i)$ over time t.
You start with v_1 dollars and fractionally invest it into d stocks according to $p_1 \in \triangle_d$.

Your portfolio at time 2 is worth $v_2 := \sum_{i=1}^{d} v_1 p_1(i) r_1(i) = v_1 \langle p_1, r_1 \rangle$ dollars

where $r_t(i) = \frac{S_{t+1}(i)}{S_t(i)} = \frac{\text{price of GOOG at time t+1}}{\text{price of GOOG at time t}}$.

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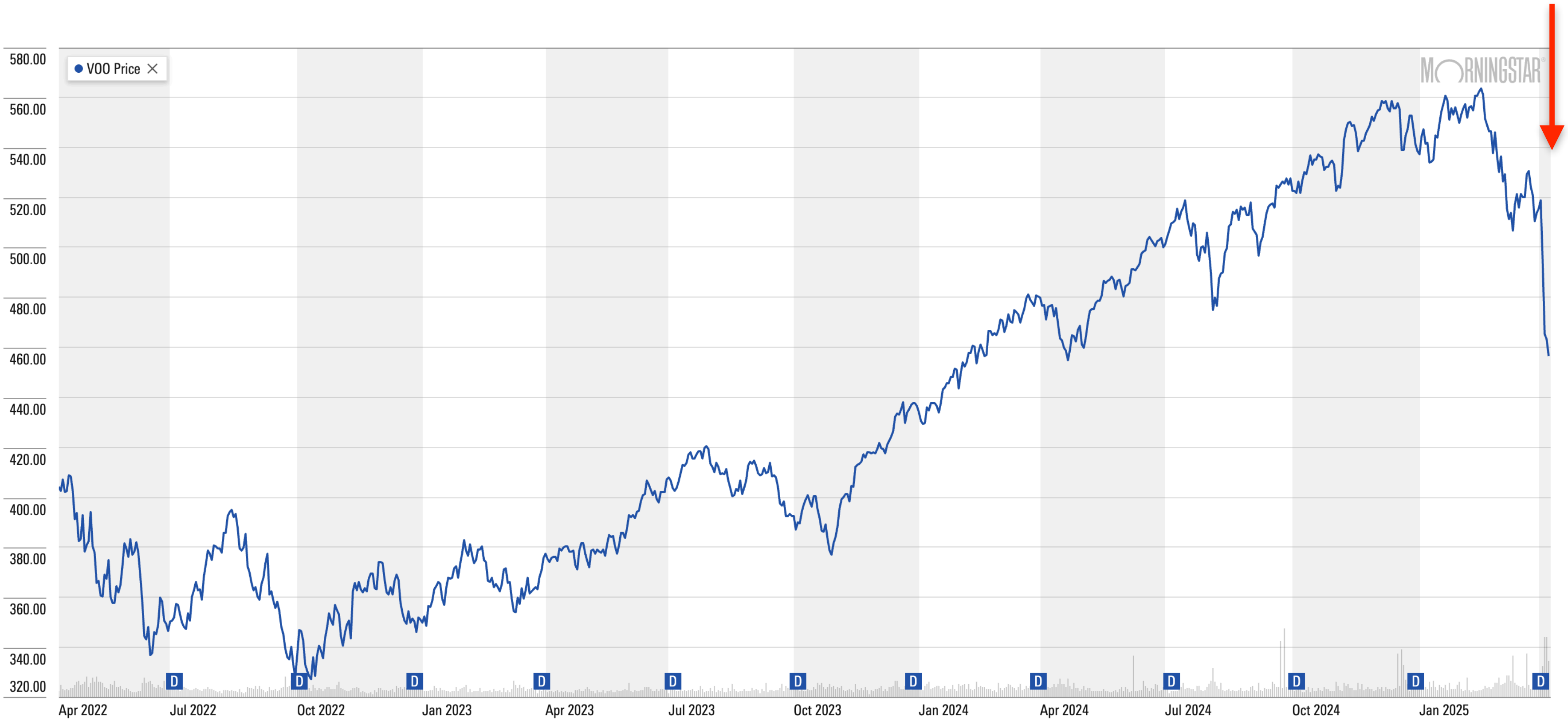
Classical Portfolio Theory (Markowitz 1952): Assume returns $r_t \in \mathbb{R}_+^n$ are IID with mean $\mu = \mathbb{E}[r_t]$ and covariance $\Sigma = \mathbb{E}[(r_t - \mu)(r_t - \mu)^\top]$. The for a return target $\bar{r} \geq 0$ solve

$$\min_{p \in \triangle_d} p^\top \Sigma p \qquad \text{subject to} \qquad p^\top \mu \geq \bar{r}$$

In practice, estimate μ, Σ from data. What could possibly go wrong?

Universal Portfolio Optimization

Trump administration
announces Tariffs



Returns are not an IID stochastic random walk!

Can we model the stock market as an online learning problem and develop an algorithm that is robust to even adversarial returns?

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After T times your portfolio is worth $v_T = v_1 \prod_{t=1}^{T-1} \langle p_t, r_t \rangle$.

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After T times your portfolio is worth $v_T = v_1 \prod_{t=1}^{T-1} \langle p_t, r_t \rangle$.

Goal: Maximize your return $\frac{v_T}{v_1}$, equivalent to $\log\left(\frac{v_T}{v_1}\right) = \sum_{t=1}^{T-1} \log \langle p_t, r_t \rangle$

Regret = $\max_{p \in \triangle_d} \sum_{t=1}^{T-1} \log \langle p, r_t \rangle - \sum_{t=1}^{T-1} \log \langle p_t, r_t \rangle$

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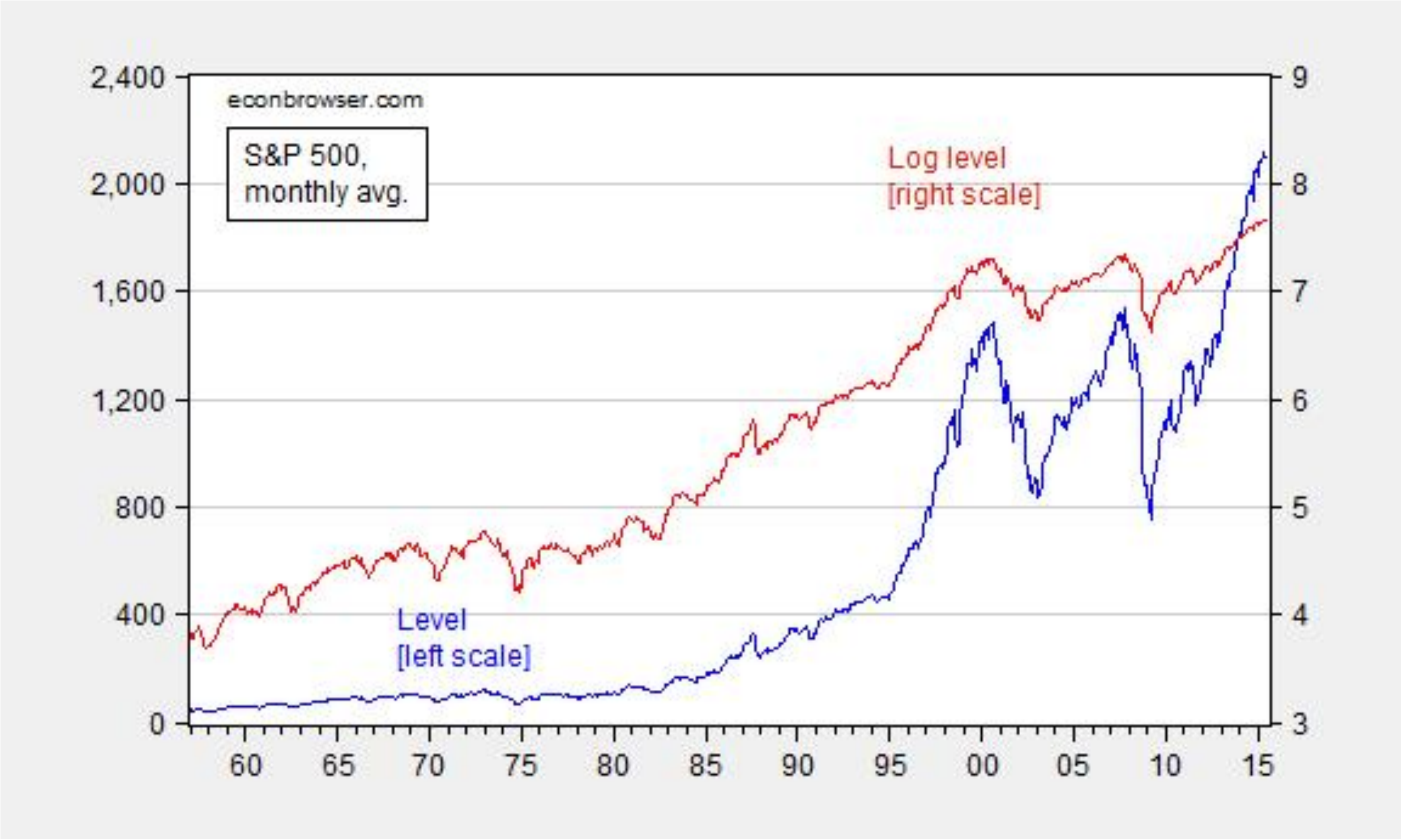
$$\text{Regret} = \max_{p \in \triangle_d} \sum_{t=1}^{T-1} \log\langle p, r_t\rangle - \sum_{t=1}^{T-1} \log\langle p_t, r_t\rangle$$

The SP500 (VOO) is an index that weights 500 stocks by their market capitalization.
An alternative index (RSP) weights these 500 stocks uniformly $p = (\frac{1}{500}, \ldots, \frac{1}{500})$.



Universal Portfolio Optimization

$$\text{Regret} = \max_{p \in \triangle_d} \sum_{t=1}^{T-1} \log\langle p, r_t \rangle - \sum_{t=1}^{T-1} \log\langle p_t, r_t \rangle$$

for $t = 1, 2, \ldots$

Player picks $p_t \in \triangle_d$

Adversary simultaneously reveals $r_t \in \mathbb{R}_+^d$

Player pays loss $\ell_t(p_t) = -\log\langle p_t, r_t \rangle$

Exponential weights algorithm

Initialize: $w_1 = (1, \ldots, 1) \in \mathbb{R}^d$

for $t = 1, 2, \ldots$

Player plays $p_t(i) = w_t(i) / \sum_{j=1}^d w_t(j)$

Adversary simultaneously reveals convex loss $\ell_t(\cdot)$

Player pays loss $\ell_t(p_t)$

Player updates weights $w_{t+1}(i) = w_t(i) \exp(-\eta \ell_t(\mathbf{e}_i))$

Universal Portfolio Optimization

$$\text{Regret} = \max_{p \in \triangle_d} \sum_{t=1}^{T-1} \log\langle p, r_t \rangle - \sum_{t=1}^{T-1} \log\langle p_t, r_t \rangle$$

Competes with the single best stock in hindsight!

Theorem: With $\eta = 1$ and $l_t(p) = -\log\langle p, r_t \rangle$, $\max_{i \in [d]} \sum_{t=1}^{T-1} \log\langle \mathbf{e}_i, r_t \rangle - \log\langle p_t, r_t \rangle \leq \log(d)$

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Proof

Theorem: With $\eta = 1$ and $l_t(p) = -\log\langle p, r_t\rangle$, $\max_{i\in[d]} \sum_{t=1}^{T-1} \log\langle \mathbf{e}_i, r_t\rangle - \log\langle p_t, r_t\rangle \leq \log(d)$

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &= \sum_{t=1}^{T} \log \frac{W_{t+1}}{W_t} \\
&= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{d} \frac{w_{t+1}(i)}{W_t}\right) \\
&= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{d} \frac{w_t(i)\exp(-\eta\ell_t(\mathbf{e}_i))}{W_t}\right) \\
&= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{d} p_t(i)\exp(-\eta\ell_t(\mathbf{e}_i))\right) \\
&= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{d} p_t(i)\exp(\log\langle \mathbf{e}_i, r_t\rangle)\right) \\
&= \sum_{t=1}^{T} \log\langle p_t, r_t\rangle
\end{aligned}$$

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &\geq \log \frac{w_{T+1}(i)}{W_1} \\
&= -\log(d) + \log\left(\prod_{t=1}^{T} \exp(-\eta\ell_t(\mathbf{e}_i))\right) \\
&= -\log(d) - \sum_{t=1}^{T} \eta\ell_t(\mathbf{e}_i) \\
&= -\log(d) + \sum_{t=1}^{T} \log\langle \mathbf{e}_i, r_t\rangle
\end{aligned}$$

$$\Longrightarrow \max_{i\in[d]} \sum_{t=1}^{T} \log\langle \mathbf{e}_i, r_t\rangle - \log\langle p_t, r_t\rangle \leq \log(d)$$

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$$\text{Regret} = \max_{p \in \triangle_d} \sum_{t=1}^{T-1} \log\langle p, r_t \rangle - \sum_{t=1}^{T-1} \log\langle p_t, r_t \rangle$$

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Is competing against <u>single best</u> stock a good benchmark? Consider just 2 stocks:

$$r_t(1) = (2, \tfrac{1}{2}, 2, \tfrac{1}{2}, 2, \tfrac{1}{2}, \ldots)$$

$$r_t(2) = (\tfrac{1}{2}, 2, \tfrac{1}{2}, 2, \tfrac{1}{2}, 2\ldots)$$

$$\prod_{t=1}^{T} \langle \mathbf{e}_i, r_t \rangle = 1$$

$$\prod_{t=1}^{T} \langle \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, r_t \rangle = \left((\tfrac{1}{2})^2 + 1 \right)^{T/2}$$

How do we compete with any $p \in \triangle_d$?

Example due to Elad Hazan's *Introduction to Online Convex Optimization*

$$\ell_t(p) = -\log(\langle p, r_t \rangle)$$

$$\nabla \ell_t(p) = -\frac{r_t}{\langle p, r_t \rangle}$$

Intuitively $r_t \in [.9, 1.1]^d$

$$\|\nabla \ell_t(p)\|_\infty \lesssim 1.1$$

$$\max_{p \in \Delta_d} \|p\|_2 = 1$$

$$\|\nabla \ell_t(p)\|_2 \lesssim 1.1 \cdot \sqrt{d}$$

$$p_{t+1} = \Pi_{\Delta_d}\left(p_t - \eta \nabla \ell_t(p_t)\right)$$

OGD: $$\max_{p \in \Delta_d} \sum_{t=1}^{T} \ell_t(p_t) - \ell_t(p) = \max_{p \in \Delta_d} \sum_{t=1}^{T} \log(\langle p, r_t \rangle) - \log(\langle p_t, r_t \rangle)$$

$$\leq R \cdot G \cdot \sqrt{T} \simeq \sqrt{dT}$$

$$z_t = \nabla \ell_t(p_t)$$

$$-\log(x)$$

If $\ell(x)$ is convex then

$$\ell(y) \geq \ell(x) + \nabla \ell(x)^T (y - x)$$

$$\Longrightarrow \ell(x) - \ell(y) \leq \nabla \ell(x)^T (x - y)$$

$$\Longrightarrow \max_{p \in \Delta_d} \sum_{t=1}^{T} \ell_t(p_t) - \ell_t(p) \leq \max_{p \in \Delta_d} \sum_{t=1}^{T} \nabla \ell(p_t)^T (p_t - p)$$

$$= \max_{i \in [d]} \sum_{t=1}^{T} \langle z_t, p_t \rangle - \langle z_t, e_i \rangle$$

$$\leq \sqrt{T \log d}$$

$$\max_{p \in \Delta_d} \sum_{t=1}^{T} \ell_t(p_t) - \ell_t(p) \leq R_T$$

$$\sum_{t=1}^{T} \ell_t(p) \leq \max_{p \in \Delta_d} \sum_{t=1}^{T} \ell_t(p) + R_T$$

$$\sum_{t=1}^{T} \log(\langle p_t, r_t \rangle) \geq \max_{p \in \Delta_d} \sum_{t=1}^{T} \log(\langle p, r_t \rangle) - R_T$$

Wealth at time $T = \prod_{t=1}^{T} \langle p_t, r_t \rangle \geq \max_{p \in \Delta_d} \prod_{t=1}^{T} \langle p, r_t \rangle \cdot \exp(-R_T)$

For any sequence $u_t \in \Delta_d \; \forall t$

$$\tilde{p}_t = (1 - \tfrac{1}{T}) p_t + \tfrac{1}{T}(\tfrac{1}{d} \mathbb{1})$$

$$\sum_{t=}^{T} \log(\langle u_t, r_t \rangle) - \log(\langle \tilde{p}_t, r_t \rangle) \leq \sqrt{T \log d} + \sqrt{\sum_{t=1}^{T} \|u_t - u_{t+1}\|_2}$$

For any $[s, t] \subset \{1, \ldots, T\}$

$$\max_{p \in \Delta_d} \sum_{t=1}^{T} \log(\langle p, r_t \rangle) - \log(\langle \tilde{p}_t, r_t \rangle) \leq \sqrt{T \log d}$$

Continuous Exponential weights

Continuous Exponential Weights

Fix a convex set $\mathscr{A}$ and a convex loss function $l(\cdot, z) : \mathscr{A} \to \mathbb{R}$ for each $z \in \mathscr{Z}$.

Theorem:

For any $\eta > 0$ and $l(\cdot, \cdot) \in [0,1]$ we have $\max_{a \in \mathscr{A}} \sum_{t=1}^{T-1} l(a_t, z_t) - l(a, z_t) \leq \frac{d \log(T)}{\eta} + \frac{\eta T}{8} + 1$

Continuous Exponential weights algorithm

Initialize: $w_1(a) = 1$ for all $a \in \mathcal{A}$

for $t = 1, 2, \ldots$

Player plays $a_t = \mathbb{E}_{A \sim p_t}[A]$ where $p_t(a) = \frac{w_t(a)}{\int_{a \in \mathcal{A}} w_t(a) da}$

Adversary simultaneously reveals z_t and convex loss $\ell(\cdot, z_t)$

Player pays loss $\ell(a_t, z_t)$

Player updates weights $w_{t+1}(a) = w_t(a) \exp(-\eta \ell(a, z_t))$

Continuous Exponential Weights

Let $\gamma = \frac{1}{T}$, $a^* \in \arg\min_{a\in\mathcal{A}} \sum_{t=1}^{T} \ell(a, z_t)$, $\mathcal{N}_\gamma\{(1-\gamma)a^* + \gamma a, a \in \mathcal{A}\}$

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &= \log\left(\frac{\int_{a\in\mathcal{A}} w_{T+1}(a)da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&\geq \log\left(\frac{\int_{a\in\mathcal{N}_\gamma} w_{T+1}(a)da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&= \log\left(\frac{\int_{a\in\mathcal{N}_\gamma} \exp\left(-\eta\sum_{t=1}^{T} \ell(a, z_t)\right) da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&= \log\left(\frac{\int_{a\in\gamma\mathcal{A}} \exp\left(-\eta\sum_{t=1}^{T} \ell((1-\gamma)a^* + a, z_t)\right) da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&\geq \log\left(\frac{\int_{a\in\mathcal{A}} \exp\left(-\eta\sum_{t=1}^{T} \ell\left((1-\gamma)a^* + \gamma a, z_t\right)\right) \gamma^d da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&\geq \log\left(\frac{\int_{a\in\mathcal{A}} \exp\left(-\eta\sum_{t=1}^{T} (\ell(a^*, z_t) + \gamma\ell(a, z_t))\right) \gamma^d da}{\int_{a\in\mathcal{A}} 1da}\right) \\
&= d\log\gamma - \eta\sum_{t=1}^{T} \ell(a^*, z_t) - \eta\gamma T
\end{aligned}$$

$$\begin{aligned}
\log \frac{W_{t+1}}{W_t} &= \log\left(\int_{\mathcal{A}} \frac{w_t(a)}{W_t}\exp(-\eta\ell(a, z_t))da\right) \\
&= \log\left(\mathbb{E}\exp(-\eta\ell(A, z_t))\right) \text{ where } \mathbb{P}(A = a) = \frac{w_t(a)}{W_t} \\
&\leq -\eta\mathbb{E}\ell(A, z_t) + \frac{\eta^2}{8} \text{ (Hoeffding's lemma)} \\
&\leq -\eta\ell(\mathbb{E}A, z_t) + \frac{\eta^2}{8} \text{ (Jensen's inequality)} \\
&= -\eta\ell(a_t, z_t) + \frac{\eta^2}{8}.
\end{aligned}$$

Proof due to Sebastien Bubeck's *Introduction to Online Optimization*

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Theorem:

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Continuous Exponential weights algorithm

Initialize: $w_1(a) = 1$ for all $a \in \mathcal{A}$

for $t = 1, 2, \ldots$

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&= \log\left(\frac{\int_{a\in\mathcal{A}} \exp(-\eta\sum_{t=1}^T \ell(a, z_t))da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&= \log\left(\frac{\int_{a\in\mathcal{A}} \prod_{t=1}^T \langle a, z_t\rangle da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&= \log\left(\frac{\int_{a\in\mathcal{A}} \prod_{t=1}^T \langle a, z_t\rangle da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&\geq \log\left(\frac{\int_{a\in\mathcal{N}_\gamma} \prod_{t=1}^T \langle a, z_t\rangle da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&\geq \log\left(\frac{\int_{a\in\gamma\mathcal{A}} \prod_{t=1}^T \langle (1-\gamma)a^* + a, z_t\rangle da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&= \log\left(\frac{\int_{a\in\mathcal{A}} \gamma^d \prod_{t=1}^T \langle (1-\gamma)a^* + \gamma a, z_t\rangle da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&\geq \log\left(\frac{\int_{a\in\mathcal{A}} \gamma^d \left((1-\gamma)\prod_{t=1}^T \langle a^*, z_t\rangle + \gamma \prod_{t=1}^T \langle a, z_t\rangle\right) da}{\int_{a\in\mathcal{A}} 1 da}\right) \\
&\geq -d\log(1/\gamma) + \log(1-\gamma) + \sum_{t=1}^T \langle a^*, z_t\rangle
\end{aligned}$$

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &= \sum_{t=1}^T \log \frac{W_{t+1}}{W_t} \\
&= \sum_{t=1}^T \log\left(\int_{\mathcal{A}} \frac{w_t(a)}{W_t} \exp(-\eta\ell(a, z_t))da\right) \\
&= \sum_{t=1}^T \log\left(\mathbb{E}_{A\sim p_t}[\exp(-\eta\ell(A, z_t))]\right) \\
&= \sum_{t=1}^T \log\langle a_t, z_t\rangle
\end{aligned}$$

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