Adversary picks $\ell_t \in [-1,1]^n \quad \forall t=1,\dots,T$

for $\quad t=1,2,\dots,T$

Player chooses $I_t \in [n]$ and observes ℓ_{t,I_t}

$$\text{Regret} = \max_i \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,I_t} - \ell_{t,i}\right]$$

Let $\quad p_t \in \Delta_n := \{ x \in \mathbb{R}^n_+ : \sum_i x_i = 1 \}, \ \mathbb{P}(I_t = i) = p_{t,i}$

$$\hat{\ell}_{t,i} = \frac{\mathbb{1}\{I_t = i\}}{p_{t,i}} \ell_{t,I_t} \qquad \forall i$$

$$\mathbb{E}[\hat{\ell}_{t,i}] = \sum_{j=1}^{n} \mathbb{P}(I_t = j) \frac{\mathbb{1}\{j=i\}}{p_{t,i}} \ell_{t,j}$$ ← only well-defined if $p_{t,i} > 0 \ \forall t,i$

$$= \sum_{j=1}^{n} \cancel{p_{t,i}} \frac{\mathbb{1}\{j=i\}}{\cancel{p_{t,i}}} \ell_{t,j}$$

$$= \ell_{t,i}$$

$$\mathbb{E}\left[\sum_{t=1}^{n} \hat{\ell}_{t,i}\right] = \sum_{t=1}^{T} \ell_{t,i}$$

$$\mathbb{V}(\hat{\ell}_{t,i}) = \mathbb{E}[\hat{\ell}_{t,i}^2] - \mathbb{E}[\hat{\ell}_{t,i}]^2$$

$$= \sum_{j=1}^{n} \mathbb{P}(I_t = j) \frac{\mathbb{1}\{j=i\}^2}{p_{t,i}^2} \ell_{t,i}^2 \quad - \quad \ell_{t,i}^2$$

$$= \sum_{j=1}^{n} \frac{\mathbb{1}\{j=i\}}{p_{t,i}} \ell_{t,i}^2 \quad - \quad \ell_{t,i}^2$$

$$= \ell_{t,i}^2 \left(\frac{1}{p_{t,i}} - 1 \right)$$

EXP3(γ)

Adversary chooses $\ell_t \in [-1,1]^n \ \forall t$

Init: Choose $\lambda \in \Delta_n$. Set $p_1 = (\frac{1}{n}, \dots, \frac{1}{n})$, $w_1 = (1, \dots, 1)$, $\eta > 0$

for $t = 1, 2, \dots, T$

Draw I_t from $q_t = (1-\gamma) p_t + \gamma \lambda$

Define $\hat{\ell}_{t,i} = \frac{\mathbb{1}\{I_t = i\}}{q_{t,i}} \ell_{t,i}$

$$w_{t+1,i} = w_{t,i} \exp(-\eta \hat{\ell}_{t,i}) \qquad p_{t+1,i} = \frac{w_{t+1,i}}{\sum_j w_{t+1,j}}$$

$$\mathbb{E}\left[\sum_{t=1}^T \ell_{t,I_t} - \ell_{t,i}\right] = \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n q_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq \gamma T + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n (1-\gamma) p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq \gamma T - \gamma \mathbb{E}\left[\sum_{t=1}^T \underbrace{\sum_{j=1}^n p_{t,j} \ell_{t,j}}_{\geq -1}\right] + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq 2\gamma T + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq 2\gamma T + \mathbb{E}\left[\sum_{t=1}^{T} \sum_{j=1}^{n} p_{t,j} \hat{\ell}_{t,j} - \hat{\ell}_{t,i} \right]$$

$$W_t = \sum_{i=1}^{n} w_{t,i}$$

$$\log\left(\frac{W_{T+1}}{W_1}\right) \geq \log\left(\frac{w_{T+1,i}}{W_1}\right) = \log(w_{T+1,i}) - \log(n)$$

$$= -\eta \sum_{t=1}^{T} \hat{\ell}_{t,i} - \log(n)$$

$$e^{x} \leq 1 + x + x^2 \quad \forall x \leq 1$$

$$1 + x \leq e^{x} \quad \forall x$$

$$\log\left(\frac{W_{T+1}}{W_1}\right) = \sum_{t=1}^{T} \log\left(\frac{W_{t+1}}{W_t}\right)$$

$$= \sum_{t=1}^{T} \log\left(\frac{1}{W_t} \sum_{i=1}^{n} w_{t,i} \exp(-\eta \hat{\ell}_{t,i})\right)$$

$$= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{n} p_{t,i} \exp(-\eta \hat{\ell}_{t,i})\right)$$

$$\leq \sum_{t=1}^{T} \log\left(1 - \sum_{i=1}^{n} p_{t,i} \eta \hat{\ell}_{t,i} + \sum_{i=1}^{n} p_{t,i} \eta^2 \hat{\ell}_{t,i}^2\right)$$

$$\leq -\eta \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i} \hat{\ell}_{t,i} + \eta^2 \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i} \hat{\ell}_{t,i}^2$$

$$\Rightarrow \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i} \hat{\ell}_{t,i} - \hat{\ell}_{t,j} \leq \frac{\log(n)}{\eta} + \eta \sum^{T} \sum^{n} p_{t,i} \hat{\ell}_{t,i}^2$$

c $\tau=1, i=1$

$$|\eta \hat{\ell}_{t,i}| \le 1$$

$$\Longrightarrow$$

$$\boxed{\max_j \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,I_t} - \ell_{t,j}\right] \le 2\gamma T + \frac{\log(n)}{\eta} + \eta\, \mathbb{E}\left[\sum_{\tau=1}^{T}\sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i}^2\right]}$$

$$q_{t,i} = \underbrace{(1-\gamma)p_{t,i} + \gamma}_{\ge \gamma\lambda_i = \gamma\cdot 1/n}\lambda_i \ge (1-\gamma)p_{t,i} \Rightarrow p_{t,i} \le \frac{q_{t,i}}{1-\gamma}$$

$$\mathbb{E}\left[\sum_{\tau=1}^{T}\sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i}^2\right] = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} p_{t,i} \frac{\mathbb{1}\{I_t=i\}}{q_{t,i}^2} \ell_{t,i}^2\right] \qquad \lambda = \left(\tfrac{1}{n}, \dots, \tfrac{1}{n}\right)$$

$$\le \frac{1}{1-\gamma} \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} \frac{\mathbb{1}\{I_t=i\}}{q_{t,i}} \ell_{t,i}^2\right]$$

$$= \frac{1}{1-\gamma} \sum_{t=1}^{T}\sum_{i=1}^{n} \ell_{t,i}^2$$

$$\le \frac{nT}{1-\gamma}$$

$$\text{Regret} \le 2\gamma T + \frac{\log(n)}{\eta} + \frac{\eta n T}{(1-\gamma)}$$

Need $-\eta\hat{\ell}_{t,i} \le 1$

$$|\eta \hat{\ell}_{t,i}| = \left|\eta \frac{\mathbb{1}\{I_t=i\}}{q_{t,i}} \ell_{t,i}\right| \le \eta \cdot \frac{1}{\gamma/n} = \frac{n\eta}{\gamma} \le 1$$

<u>if</u> $\gamma = n\eta$

$$\text{Regret} \leq 2n\eta T + \frac{\log(n)}{\eta} + \frac{\eta n T}{1-n\eta}$$

$$\leq c\sqrt{nT\log(n)}$$

$$\eta = \sqrt{\frac{\log(n)}{3nT}} \wedge \frac{1}{2n}$$

FTPL - follow the perturbed leader

$$\sum_{s=1}^{t} \hat{l}_{s,i}$$

$$I_t = \operatorname{argmax}_i Z_i + \eta \sum_{s=1}^{t} \hat{l}_{t,i}$$

Adversarial Linear Bandits

Input arm set $\mathcal{X} \subset \mathbb{R}^d$

Adversary chooses $\ell_t \in \mathbb{R}^d$ $\forall t = 1, \dots, T$ arbitrarily

s.t. $\max_{x \in \mathcal{X}} |\langle x, \ell_t \rangle| \leq 1$ $\forall t$

for $t = 1, 2, \dots$

Player chooses $x_t \in \mathcal{X}$

Gets loss $\langle x_t, \ell_t \rangle$

Regret $\max_{x \in \mathcal{X}} \sum_{t=1}^{T} \langle x - x_t, \ell_t \rangle$

$\mathcal{X} = (x_1, \dots, x_n)$ $\qquad$ $I_t \sim q_t$ $\quad$ $x_{I_t} \equiv x_t$

For $\underline{x_i \in \mathcal{X}}$, define $\hat{\ell}_{t,i} = x_i^T A(q_t)^{-1} x_t x_t^T \ell_t$

$$A(q_t) = \sum_{x \in \mathcal{X}} q_{t,x}\, x x^T \equiv \sum_{i=1}^{n} q_{t,i}\, x_i x_i^T$$

$$\mathbb{E}[\hat{\ell}_{t,i}] = x_i^T A(q_t)^{-1} \underbrace{\mathbb{E}[x_t x_t^T]}_{= A(q_t)} \ell_t$$

$$= x_i^T \ell_t = \ell_{t,i}$$

$$\mathbb{E}[\hat{\ell}_{t,i}^2] = x_i^T A(q_t)^{-1} \mathbb{E}[x_t x_t^T \underline{(x_t^T \ell_t)^2}] A(q_t)^{-1} x_i$$

$$\leq x_i^T A(q_t)^{-1} x_i$$

Want to minimize $\max_{x \in \mathcal{X}} x^T A(q_t)^{-1} x = \max_{x \in \mathcal{X}} \|x\|^2_{A(q_t)^{-1}}$

Recall $\min_{\lambda \in \Delta_{\mathcal{X}}} \max_{x \in \mathcal{X}} x^T A(\lambda)^{-1} x = d$ by Kiefer-Wolfowitz.

To apply EXP3(γ) theorem we need to show $|\eta \hat{\ell}_{t,i}| \leq 1 \quad \forall t, i$

$$|\eta \hat{\ell}_{t,i}| \leq \eta |x_i^T A(q_t)^{-1} x_t| \underbrace{|x_t^T \ell_t|}_{\leq 1}$$

$$\underbrace{q_t = (1-\gamma) p_t + \gamma \lambda}_{\lambda}$$

G-opt design

$$\leq \eta |x_i^T A(q_t)^{-1} x_t|$$

$$\leq \eta \|x_i\|_{A(q_t)^{-1}} \cdot \|x_t\|_{A(q_t)^{-1}}$$

$$\leq \frac{\eta}{\gamma} \|x_i\|_{A(\lambda)^{-1}} \cdot \|x_t\|_{A(\lambda)^{-1}}$$

$$\leq \frac{\eta d}{\gamma} \Rightarrow \gamma = \eta d \Rightarrow |\eta \hat{\ell}_{t,i}| \leq \quad \forall t, i$$

$$\mathbb{E}\left[\sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i} \hat{\ell}_{t,i}^2\right] = \mathbb{E}\left[\sum_t \sum_i p_{t,i}\, x_i^T A(q_t)^{-1} x_t x_t^T A(q_t)^{-1} x_i \underbrace{(x_t^T \ell_t)^2}_{\leq 1}\right]$$

$$= \sum_t \sum_i p_{t,i} \, x_i^T A(q_t)^{-1} \mathbb{E}[x_t x_t^T] A(q_t)^{-1} x_i$$

Trace (ABC)
$= Tr(BCA)$
$= Tr(CAB)$

$$= \sum_t \sum_i p_{t,i} \, x_i^T A(q_t)^{-1} x_i$$

$$\leq \frac{1}{1-\gamma} \sum_t \sum_i p_{t,i} \, x_i^T A(p_t)^{-1} x_i$$

$$= \frac{1}{1-\gamma} \sum_t Tr\Big(\underbrace{\sum_i p_{t,i} x_i x_i^T}_{=A(p_t)} A(p_t)^{-1} \Big)$$

$$= \frac{dT}{1-\gamma}$$

$\Longrightarrow$ (By above analogy to MAB)

Regret $\leq c\sqrt{dT \log d}$.