Adversary picks $\ell_t \in [-1,1]^n \ \forall t=1,\dots,T$

for $t=1,2,\dots,T$

Player chooses $I_t \in [n]$ and observes ℓ_{t,I_t}

$$\text{Regret} = \max_i \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,I_t} - \ell_{t,i}\right]$$

Let $p_t \in \Delta_n = \{x \in \mathbb{R}^n_+ : \sum_i x_i = 1\}$, $\mathbb{P}(I_t = i) = p_{t,i}$

$$\hat{\ell}_{t,i} = \frac{\mathbb{1}\{I_t = i\}}{p_{t,i}} \ell_{t,I_t} \qquad \forall i$$

$$\mathbb{E}[\hat{\ell}_{t,i}] = \sum_{j=1}^{n} \mathbb{P}(I_t = j) \frac{\mathbb{1}\{j=i\}}{p_{t,i}} \ell_{t,j}$$ ← only well-defined if $p_{t,i} > 0 \ \forall t,i$

$$= \sum_{j=1}^{n} \cancel{p_{t,i}} \frac{\mathbb{1}\{j=i\}}{\cancel{p_{t,i}}} \ell_{t,j}$$

$$= \ell_{t,i}$$

$$\mathbb{E}\left[\sum_{t=1}^{n} \hat{\ell}_{t,i}\right] = \sum_{t=1}^{T} \ell_{t,i}$$

$$\mathbb{V}(\hat{\ell}_{t,i}) = \mathbb{E}[\hat{\ell}_{t,i}^2] - \mathbb{E}[\hat{\ell}_{t,i}]^2$$

$$= \sum_{j=1}^{n} \mathbb{P}(I_t = j) \frac{\mathbb{1}\{j=i\}^2}{p_{t,i}^2} \ell_{t,i}^2 - \ell_{t,i}^2$$

$$= \sum_{j=1}^{n} \frac{\mathbb{1}\{j=i\}}{p_{t,i}} \ell_{t,i}^2 - \ell_{t,i}^2$$

$$= \ell_{t,i}^2 \left(\frac{1}{p_{t,i}} - 1 \right)$$

EXP3 (γ)

Adversary chooses $\ell_t \in [-1,1]^n \ \forall t$

Init: Choose $\lambda \in \Delta_n$. Set $p_1 = (\frac{1}{n}, \dots, \frac{1}{n})$, $w_1 = (1, \dots, 1)$, $\eta > 0$

for $t = 1, 2, \dots, T$

Draw I_t from $q_t = (1-\gamma) p_t + \gamma \lambda$

Define $\hat{\ell}_{t,i} = \frac{\mathbb{1}\{I_t = i\}}{q_{t,i}} \ell_{t,i}$

$$w_{t+1,i} = w_{t,i} \exp(-\eta \hat{\ell}_{t,i}) \qquad p_{t+1,i} = \frac{w_{t+1,i}}{\sum_j w_{t+1,j}}$$

$$\mathbb{E}\left[\sum_{t=1}^T \ell_{t,I_t} - \ell_{t,i}\right] = \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n q_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq \gamma T + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n (1-\gamma) p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq \gamma T - \gamma \mathbb{E}\left[\sum_{t=1}^T \underbrace{\sum_{j=1}^n p_{t,j} \ell_{t,j}}_{\geq -1}\right] + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq 2\gamma T + \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^n p_{t,j} \ell_{t,j} - \ell_{t,i}\right]$$

$$\leq 2\gamma T + \mathbb{E}\left[\sum_{t=1}^{T} \sum_{j=1}^{n} p_{t,j}\hat{\ell}_{t,j} - \hat{\ell}_{t,i} \right]$$

$$W_t = \sum_{i=1}^{n} w_{t,i}$$

$$\log\left(\frac{W_{T+1}}{W_1}\right) \geq \log\left(\frac{w_{T+1,i}}{W_1}\right) = \log(w_{T+1,i}) - \log(n)$$

$$e^{x} \leq 1 + x + x^2 \quad \forall x \leq 1$$

$$= -\eta \sum_{t=1}^{T} \hat{\ell}_{t,i} - \log(n)$$

$$1 + x \leq e^{x} \quad \forall x$$

$$\log\left(\frac{W_{T+1}}{W_1}\right) = \sum_{t=1}^{T} \log\left(\frac{W_{t+1}}{W_t}\right)$$

$$= \sum_{t=1}^{T} \log\left(\frac{1}{W_t} \sum_{i=1}^{n} w_{t,i} \exp(-\eta \hat{\ell}_{t,i})\right)$$

$$= \sum_{t=1}^{T} \log\left(\sum_{i=1}^{n} p_{t,i} \exp(-\eta \hat{\ell}_{t,i})\right)$$

$$\leq \sum_{t=1}^{T} \log\left(1 - \sum_{i=1}^{n} p_{t,i}\eta\hat{\ell}_{t,i} + \sum_{i=1}^{n} p_{t,i}\eta^2\hat{\ell}_{t,i}^2\right)$$

$$\leq -\eta \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i} + \eta^2 \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i}^2$$

$$\Rightarrow \sum_{t=1}^{T} \sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i} - \hat{\ell}_{t,j} \leq \frac{\log(n)}{\eta} + \eta \sum^{T} \sum^{n} p_{t,i}\hat{\ell}_{t,i}^2$$

$\tau=1, i=1$

$\Longrightarrow$

$$\max_j \mathbb{E}\left[\sum_{t=1}^{T} \ell_{t,I_t} - \ell_{t,j}\right] \le 2\gamma T + \frac{\log(n)}{\eta} + \eta\, \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i}^2\right]$$

$$q_{t,i} = \underbrace{(1-\gamma)p_{t,i} + \gamma\lambda_i}_{\ge \gamma\lambda_i = \gamma\cdot 1/n} \ge (1-\gamma)p_{t,i} \Rightarrow p_{t,i} \le \frac{q_{t,i}}{1-\gamma}$$

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} p_{t,i}\hat{\ell}_{t,i}^2\right] = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} p_{t,i}\frac{\mathbb{1}\{I_t=i\}}{q_{t,i}^2}\ell_{t,i}^2\right] \qquad \lambda = \left(\tfrac{1}{n}, \dots, \tfrac{1}{n}\right)$$

$$\le \frac{1}{1-\gamma}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} \frac{\mathbb{1}\{I_t=i\}}{q_{t,i}}\ell_{t,i}^2\right]$$

$$= \frac{1}{1-\gamma}\sum_{t=1}^{T}\sum_{i=1}^{n}\ell_{t,i}^2$$

$$\le \frac{nT}{1-\gamma}$$

$$\text{Regret} \le 2\gamma T + \frac{\log(n)}{\eta} + \frac{\eta n T}{(1-\gamma)}$$

Need $-\eta\hat{\ell}_{t,i} \le 1$

$$\left|\eta\hat{\ell}_{t,i}\right| = \left|\eta\frac{\mathbb{1}\{I_t=i\}}{q_{t,i}}\ell_{t,i}\right| \le \eta\cdot\frac{1}{\gamma/n} = \frac{n\eta}{\gamma} \le 1$$

<u>if</u> $\gamma = n\eta$

$$\text{Regret} \overset{?}{\leq} 2n\eta T + \frac{\log(n)}{\eta} + \frac{\eta n T}{1-n\eta}$$

$$\leq c\sqrt{nT\log(n)}$$

$$\eta = \sqrt{\frac{\log(n)}{3nT}} \wedge \frac{1}{2n}$$

FTPL - follow the perturbed leader

$$\sum_{s=1}^{t} \hat{\ell}_{s,i}$$

$$I_t = \operatorname{argmax}_i \; Z_i + \eta \sum_{s=1}^{t} \hat{\ell}_{t,i}$$