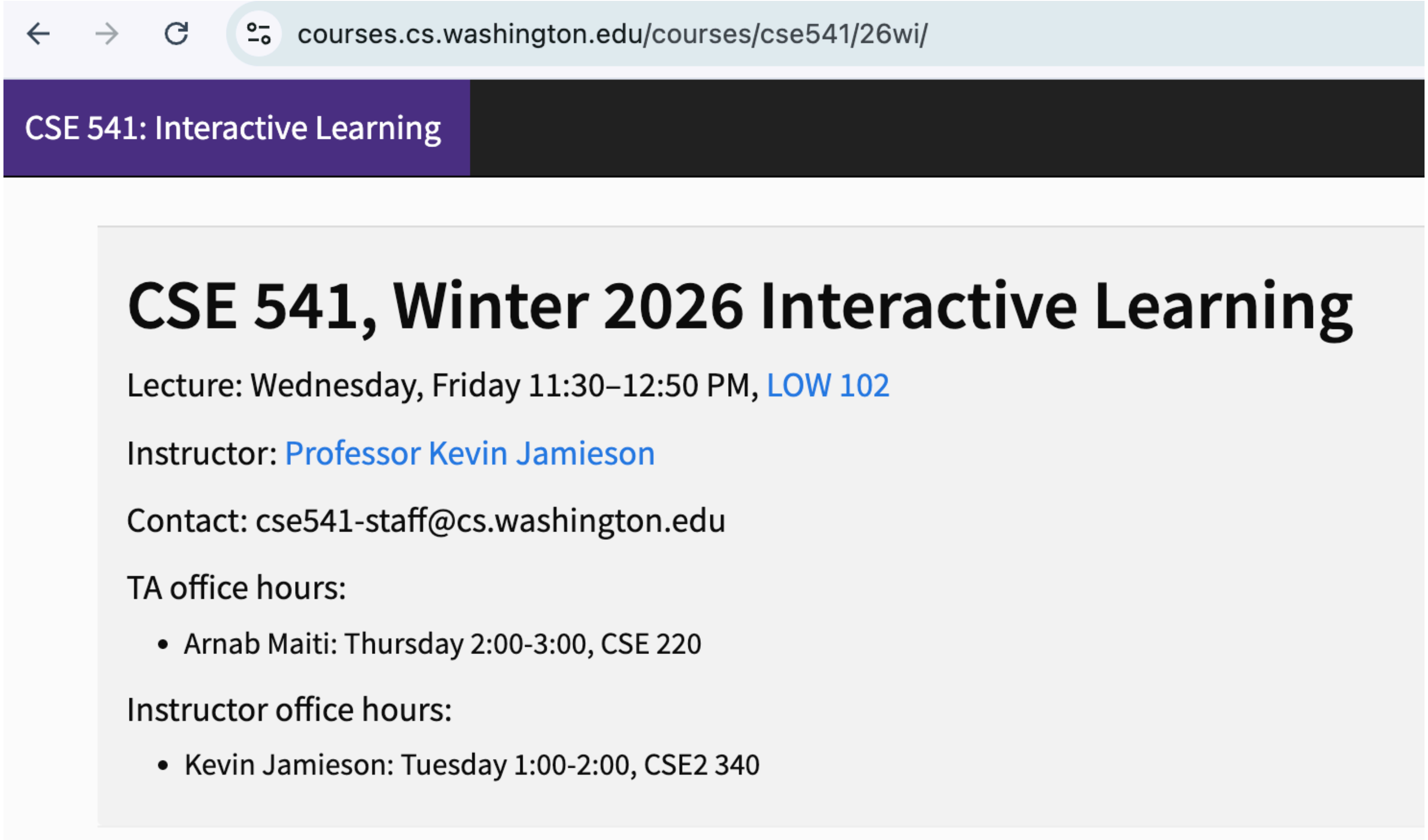
CSE 541: Interactive Learning

Kevin Jamieson



courses.cs.washington.edu/courses/cse541/26wi/

CSE 541: Interactive Learning

CSE 541, Winter 2026 Interactive Learning

Lecture: Wednesday, Friday 11:30–12:50 PM, LOW 102

Instructor: Professor Kevin Jamieson

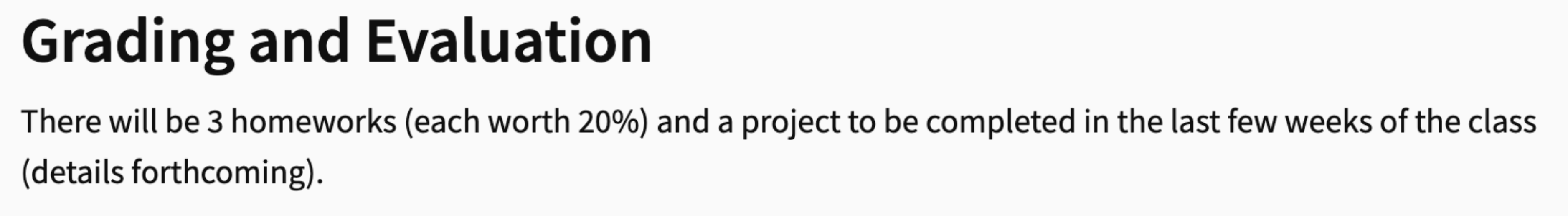
Contact: cse541-staff@cs.washington.edu

TA office hours:

- Arnab Maiti: Thursday 2:00-3:00, CSE 220

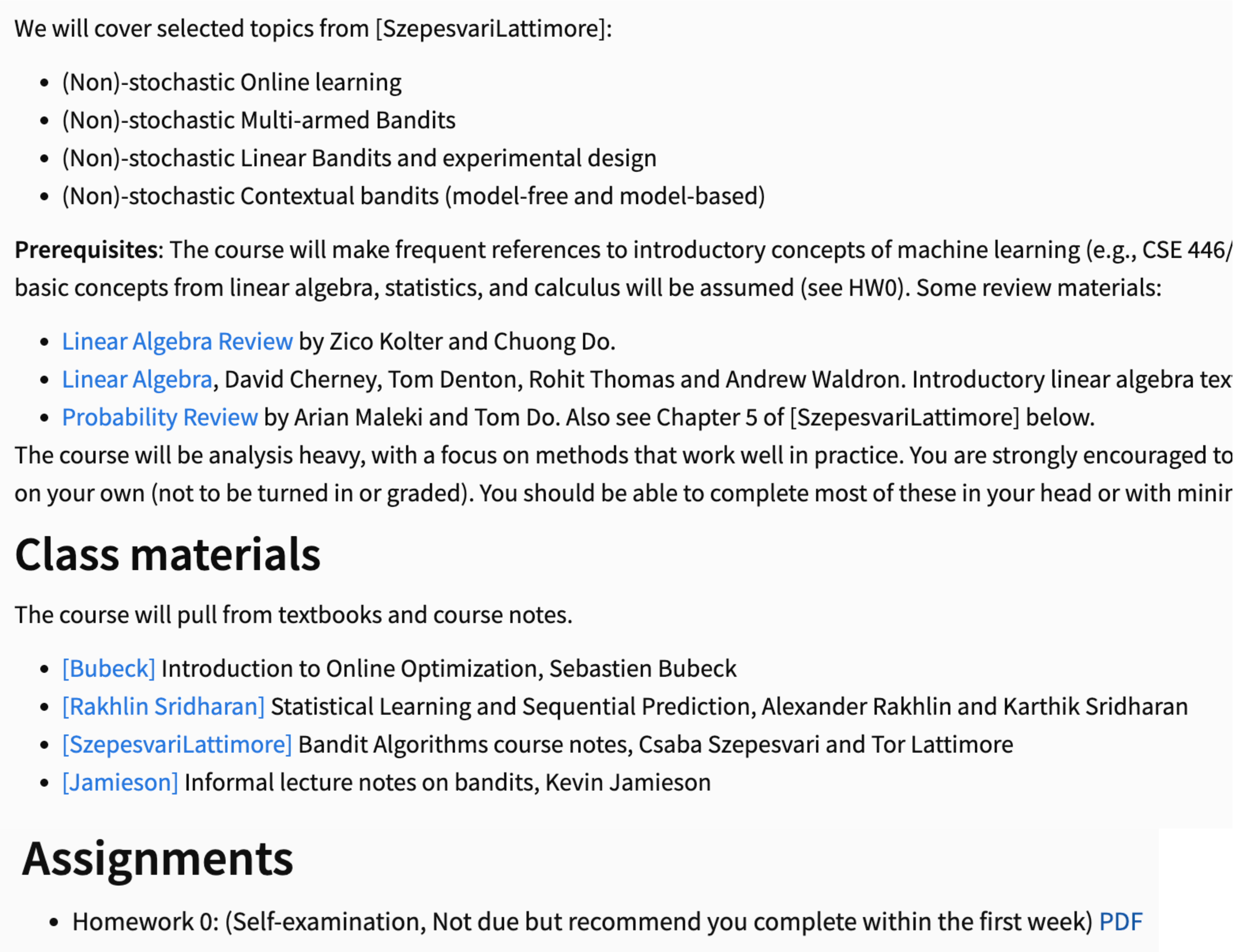
Instructor office hours:

- Kevin Jamieson: Tuesday 1:00-2:00, CSE2 340



Grading and Evaluation

There will be 3 homeworks (each worth 20%) and a project to be completed in the last few weeks of the class (details forthcoming).



We will cover selected topics from [SzepesvariLattimore]:

- (Non)-stochastic Online learning
- (Non)-stochastic Multi-armed Bandits
- (Non)-stochastic Linear Bandits and experimental design
- (Non)-stochastic Contextual bandits (model-free and model-based)

Prerequisites: The course will make frequent references to introductory concepts of machine learning (e.g., CSE 446/ basic concepts from linear algebra, statistics, and calculus will be assumed (see HW0). Some review materials:

- Linear Algebra Review by Zico Kolter and Chuong Do.
- Linear Algebra, David Cherney, Tom Denton, Rohit Thomas and Andrew Waldron. Introductory linear algebra tex
- Probability Review by Arian Maleki and Tom Do. Also see Chapter 5 of [SzepesvariLattimore] below.

The course will be analysis heavy, with a focus on methods that work well in practice. You are strongly encouraged to on your own (not to be turned in or graded). You should be able to complete most of these in your head or with minir

Class materials

The course will pull from textbooks and course notes.

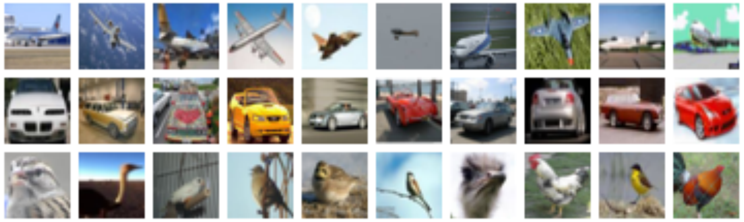
- [Bubeck] Introduction to Online Optimization, Sebastien Bubeck
- [Rakhlin Sridharan] Statistical Learning and Sequential Prediction, Alexander Rakhlin and Karthik Sridharan
- [SzepesvariLattimore] Bandit Algorithms course notes, Csaba Szepesvari and Tor Lattimore
- [Jamieson] Informal lecture notes on bandits, Kevin Jamieson

Assignments

- Homework 0: (Self-examination, Not due but recommend you complete within the first week) PDF

Standard Machine Learning Paradigm

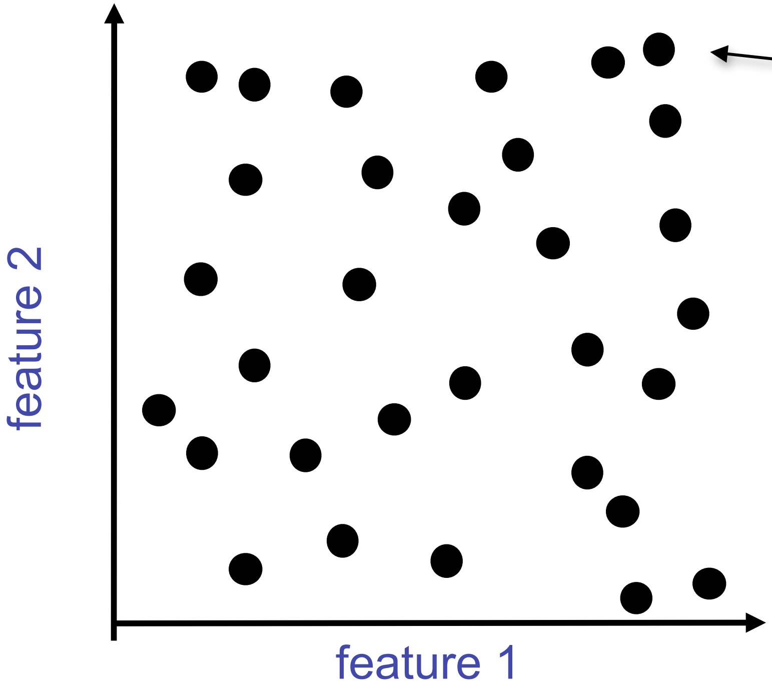
- **Data**: past observations
- **Hypotheses/Models**: devised to capture the patterns in data
- **Prediction**: apply model to forecast future observations



airplane ●

other ●

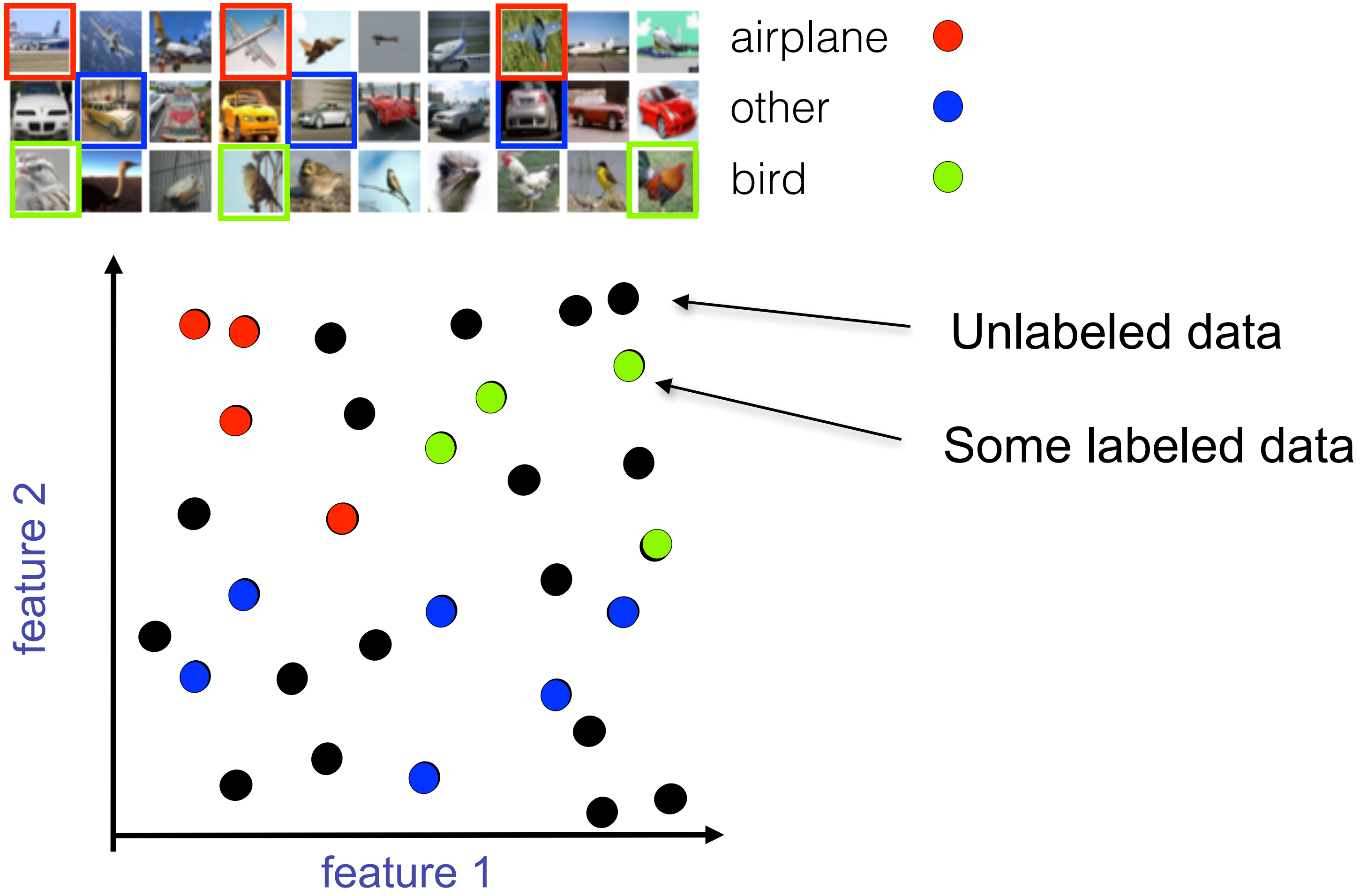
bird ●



Unlabeled data

Standard Machine Learning Paradigm

- **Data**: past observations
- **Hypotheses/Models**: devised to capture the patterns in data
- **Prediction**: apply model to forecast future observations



airplane ●

other ●

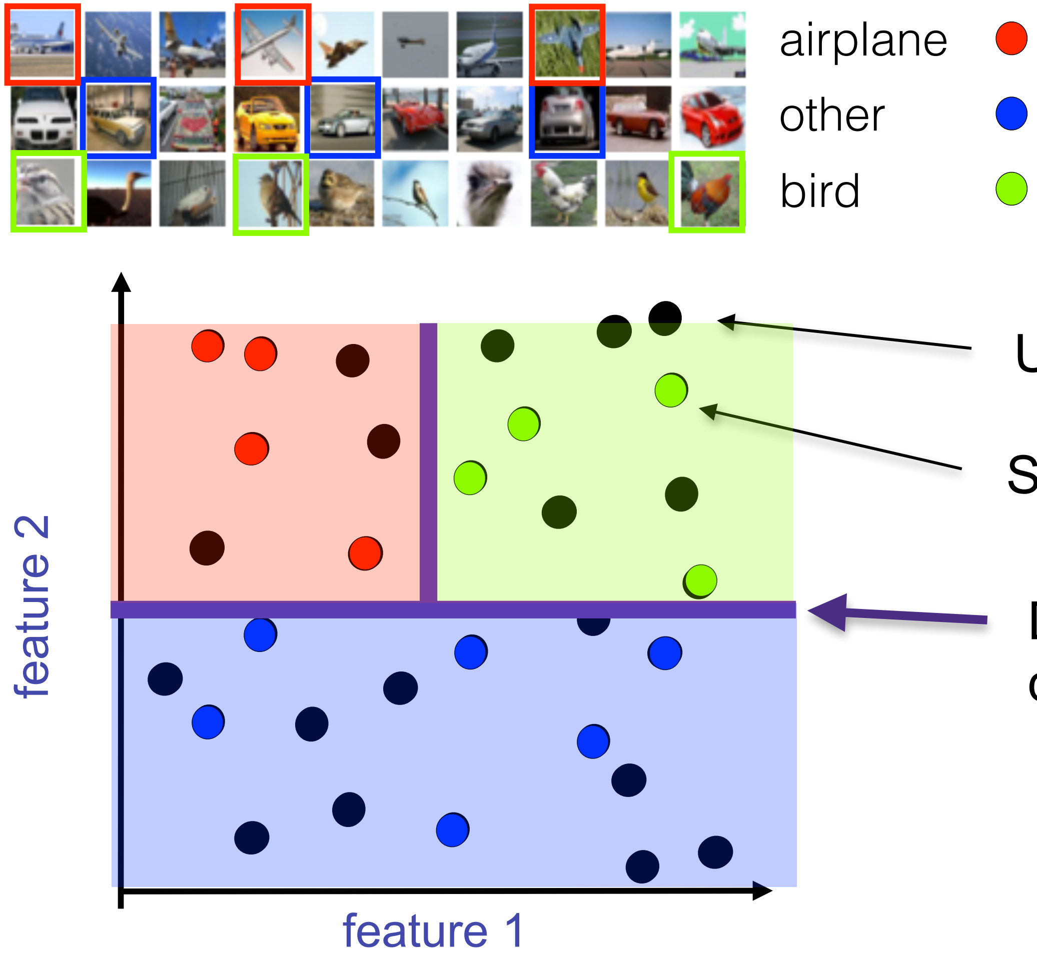
bird ●

Unlabeled data

Some labeled data

Standard Machine Learning Paradigm

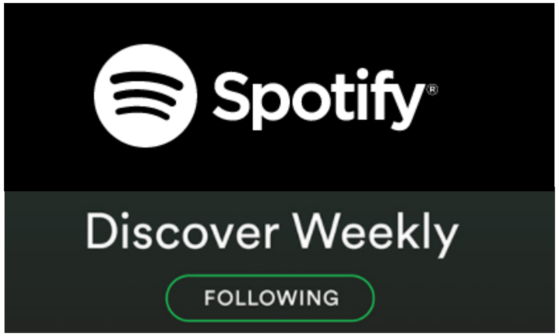
- **Data**: past observations
- **Hypotheses/Models**: devised to capture the patterns in data
- **Prediction**: apply model to forecast future observations



Unlabeled data

Some labeled data

Decision rule learned based on labeled data and applied to future unlabeled data



You may also like…

Do these applications actually fall into the standard machine learning paradigm?

Generalization Bounds

Realizable case

Fix a finite hypothesis class $\mathscr{H} = \{h_1, h_2, \ldots, \}$ where $h(x) \in \{-1,1\}$.

You are given a data set $(x_1, y_1), \ldots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i = h_*(x_i)$ for some $h_* \in \mathscr{H}$

Let $\widehat{h} = \arg\min_{h \in \mathscr{H}} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{h(x_i) \neq y_i\}$ how "good" is $\widehat{h}$?

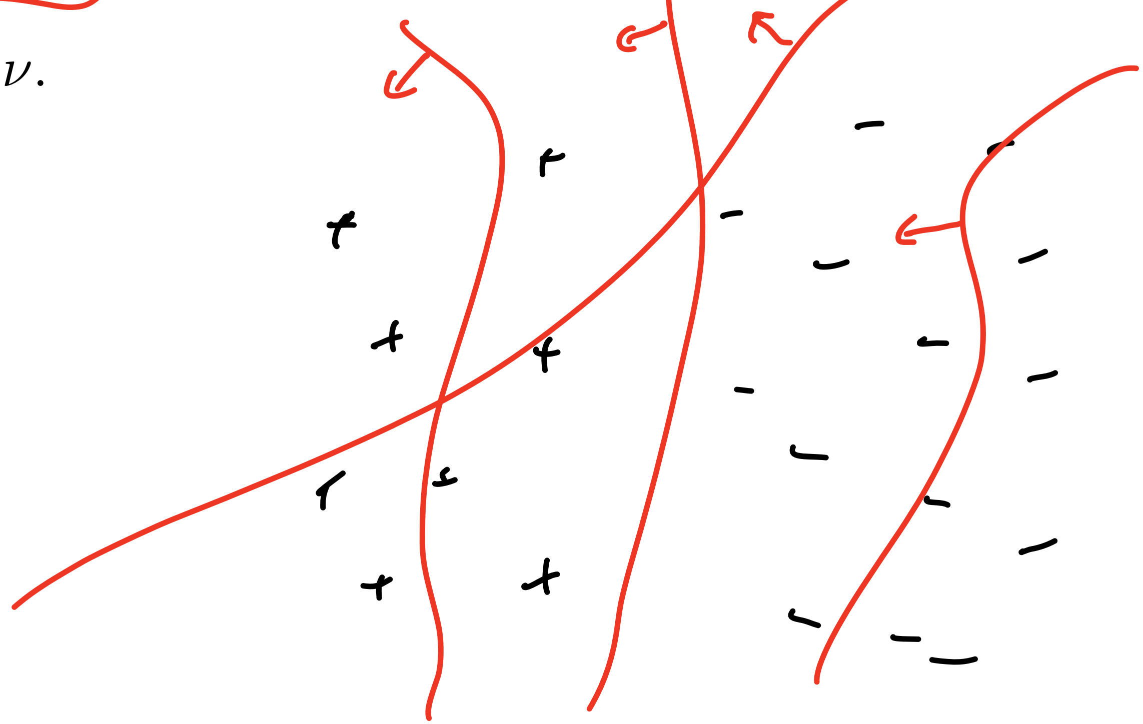
Realizable case

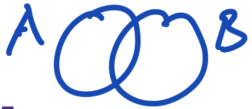
Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. Assume there exists an $h_* \in \mathcal{H}$ such that $\underline{R(h_*) = 0}$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$\boxed{\hat{R}_n(\hat{h}) = 0}$$

$$R(\widehat{h}) \leq \frac{\log(|\mathcal{H}|/\delta)}{n}$$

where $(X, Y) \sim \nu$.





$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$\le \mathbb{P}(A) + \mathbb{P}(B)$$

Realizable case - Proof

$$\mathbb{P}\left(R(\hat{h}) > \varepsilon \right) = \mathbb{P}\left(\left\{ R(\hat{h}) > \varepsilon, \bigcap_{i=1}^{n} \{\hat{h}(x_i) = y_i\} \right\} \right)$$

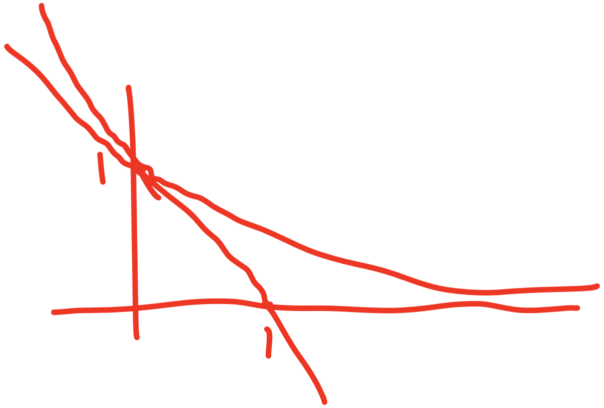
$$\le \mathbb{P}\left(\bigcup_{h \in \mathcal{H}} \left\{ R(h) > \varepsilon, \bigcap_{i=1}^{n} \{h(x_i) = y_i\} \right\} \right)$$

$$\le \sum_{h \in \mathcal{H}} \mathbb{P}\left(R(h) > \varepsilon, \bigcap_{i=1}^{n} \{h(x_i) = y_i\} \right)$$

$$= \sum_{h \in \mathcal{H}} \prod_{i=1}^{n} \underline{\mathbb{P}\left(R(h) > \varepsilon, h(x_i) = y_i \right)}$$

$$\le \sum_{h \in \mathcal{H}} \prod_{i=1}^{n} (1 - \varepsilon)$$

$$= |\mathcal{H}| (1 - \varepsilon)^n \le |\mathcal{H}| e^{-n\varepsilon} = \delta$$



$$\underbrace{1 - x \le e^{-x}}_{}$$

Realizable case - Proof

Union bound: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

$$\begin{aligned}
\mathbb{P}(R(\widehat{h}) \geq \epsilon) &= \mathbb{P}(R(\widehat{h}) \geq \epsilon) \\
&= \mathbb{P}(R(\widehat{h}) \geq \epsilon \text{ and } \cap_{i=1}^{n} \{\widehat{h}(x_i) = y_i\}) \\
&\leq \mathbb{P}(\bigcup_{h \in \mathcal{H}} \left\{ R(h) \geq \epsilon \text{ and } \cap_{i=1}^{n} \{h(x_i) = y_i\} \right\}) \\
&\leq \sum_{h \in \mathcal{H}} \mathbb{P}(R(h) \geq \epsilon \text{ and } \cap_{i=1}^{n} \{h(x_i) = y_i\}) \\
&\leq \sum_{h \in \mathcal{H}} (1-\epsilon)^n \\
&\leq |\mathcal{H}| \exp(-n\epsilon)
\end{aligned}$$

$$\exp(-x) \geq (1-x) \quad \forall x$$

Realizable case

PAC = Probably Approximately Correct

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. Assume there exists an $h_* \in \mathcal{H}$ such that $R(h_*) = 0$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) \leq \frac{\log(|\mathcal{H}|/\delta)}{n}$$

where $(X, Y) \sim \nu$.

Corollary Under the conditions of the theorem (i.e., there exists an $h_* \in \mathcal{H}$ such that $R(h_*) = 0$, $(x_i, y_i) \overset{iid}{\sim} \nu$, and $\widehat{h} = \arg\min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$) we have $\mathbb{E}[R(\widehat{h})] \leq \int_{\epsilon=0}^d \mathbb{P}(R(\widehat{h}) \geq \epsilon) \leq \frac{2\log(|\mathcal{H}|)}{n}$

Agnostic (Non-realizable) case

$h_* = \arg\min_{h \in \mathcal{H}} R(h)$

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) - R(h_*) \leq \sqrt{\frac{2\log(|\mathcal{H}|/\delta)}{n}}.$$

$$R(\hat{h}) - R(h_*) = R(\hat{h}) - \hat{R}(\hat{h}) + \underbrace{\hat{R}(\hat{h}) - \hat{R}(h_*)}_{\leq 0} + \hat{R}(h_*) - R(h_*)$$

$$\leq \left(\max_{h \in \mathcal{H}} R(h) - \hat{R}_n(h) \right) + \left(\max_{h \in \mathcal{H}} \hat{R}(h) - R(h) \right)$$

$$\mathbb{P}\left(R(h) - \hat{R}(h) > \varepsilon\right) = \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{P}(h(x) \neq Y) - \mathbb{1}\{h(x_i) \neq y_i\}\right) > \varepsilon\right)$$

$$\leq \exp\left(-2n\varepsilon^2\right)$$

$$\mathbb{P}\left(\hat{R}(h) - R(h) > \varepsilon\right) \leq \exp\left(-2n\varepsilon^2\right)$$

$$\mathbb{P}\left(\max_{h \in \mathcal{H}} R(h) - \hat{R}(h) > \varepsilon\right)$$

$$= \mathbb{P}\left(\bigcup_{h \in \mathcal{H}} \{R(h) - \hat{R}(h) > \varepsilon\}\right)$$

$$\leq \sum_{h \in \mathcal{H}} \mathbb{P}\left(R(h) - \hat{R}(h) > \varepsilon\right)$$

$$\leq |\mathcal{H}| \exp\left(-2n\varepsilon^2\right) = \delta$$

Agnostic (Non-realizable) case - Proof

Lemma (Hoeffding's inequality): Let $Z_1, \dots, Z_n \overset{iid}{\sim} \nu$ where $\mathbb{E}[Z_i] = \mu$ and $Z_i \in [a, b]$ almost surely. Then

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i \geq \mu + \epsilon\right) \leq \exp\left(\frac{-2n\epsilon^2}{|b-a|^2}\right).$$

Lemma (Hoeffding's Lemma). Let X be a real-valued random variable such that $X \in [a, b]$ almost surely, and let $\mathbb{E}[X] = \mu$. Then, for any $t \in \mathbb{R}$,

$$\mathbb{E}\left[e^{t(X-\mu)}\right] \leq \exp\left(\frac{t^2(b-a)^2}{8}\right)$$

Agnostic (Non-realizable) case - Proof

Agnostic (Non-realizable) case

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \ldots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) - R(h_*) \leq \sqrt{\frac{2\log(|\mathcal{H}|/\delta)}{n}}.$$

Corollary Under the conditions of the theorem (i.e., $(x_i, y_i) \overset{iid}{\sim} \nu$, and $\widehat{h} = \arg\min_{h \in \mathcal{H}} \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$) and $|\mathcal{H}| \geq n$, we have $\mathbb{E}[R(\widehat{h})] - R(h_*) \leq \sqrt{\frac{8\log(|\mathcal{H}|)}{n}}$

Agnostic (Non-realizable) case - Interpolation

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) - R(h_*) \leq \sqrt{\frac{2R(h_*)\log(2|\mathcal{H}|/\delta)}{n}} + \frac{\log(2|\mathcal{H}|/\delta)}{n}.$$

Proof: Use Bernstein's inequality instead of Hoeffding. ■

Infinite classes

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) - R(h_*) \leq \sqrt{\frac{2R(h_*) \log(2|\mathcal{H}|/\delta)}{n}} + \frac{\log(2|\mathcal{H}|/\delta)}{n}.$$

What if $|\mathcal{H}|$ is *infinite* such as the space of all hyperplane classifers?

Infinite classes

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) - R(h_*) \leq \sqrt{\frac{2R(h_*) \log(2|\mathcal{H}|/\delta)}{n}} + \frac{\log(2|\mathcal{H}|/\delta)}{n}.$$

What if $|\mathcal{H}|$ is *infinite* such as the space of all hyperplane classifers?

Lots of tools to address this:
- minimum description length
- VC-dimension and Rademacher complexity
- Covering number / log-entropy bounds

Online Learning

UNIVERSITY *of* WASHINGTON

Realizable case

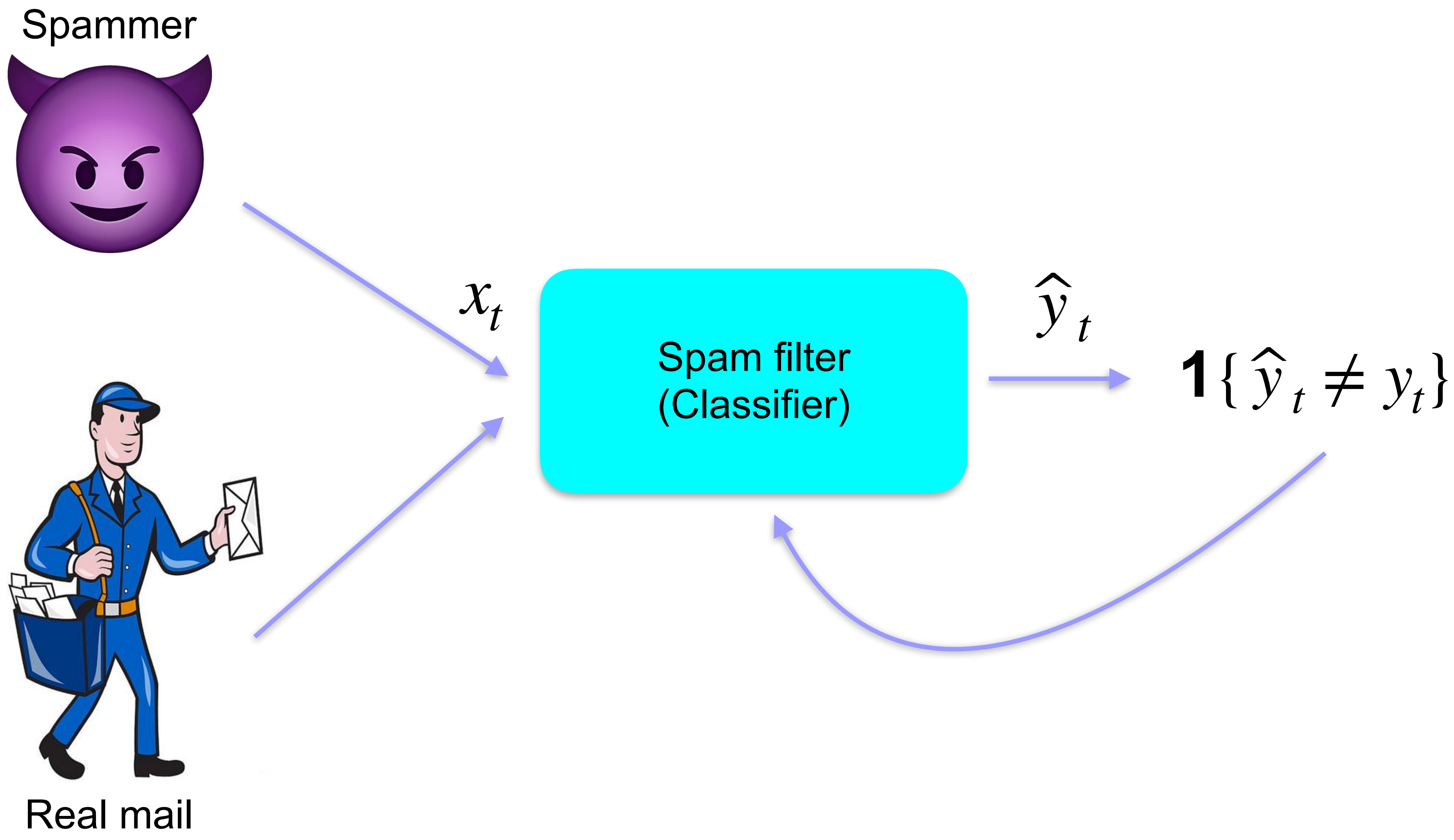
Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n) \overset{iid}{\sim} \nu$ where $y_i \in \{-1, 1\}$. For any $h \in \mathcal{H}$ define $\widehat{R}_n(h) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$ and $R(h) = \mathbb{P}(h(X) \neq Y)$ where $(X, Y) \sim \nu$. Assume there exists an $h_* \in \mathcal{H}$ such that $R(h_*) = 0$. If $\widehat{h} = \arg\min_{h \in \mathcal{H}} \widehat{R}_n(h)$ then with probability at least $1 - \delta$ we have

$$R(\widehat{h}) \leq \frac{\log(|\mathcal{H}|/\delta)}{n}$$

where $(X, Y) \sim \nu$.

All the guarantees of the previous section (and the entirety of this class so far) has relied critically on *(x,y)* being drawn **IID**. Can we say anything if *(x,y)* are chosen **adversarially**?

Online learning



Spammer

x_t

Spam filter
(Classifier)

$\hat{y}_t$

$\mathbf{1}\{\hat{y}_t \neq y_t\}$

Real mail

Online learning

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes
$\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\}$

Settings of interest:

IID $(x_t, y_t) \sim \nu$

Adversarial (x_t, y_t) arbitrary

Online learning - Realizable IID

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes
$\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\}$

IID $\quad (x_t, y_t) \sim \nu \quad y_t = h_*(x_t)$

We know learning theory! Choose $h_t \in \arg\min_{h \in \mathcal{H}} \sum_{s=1}^{t-1} \mathbf{1}\{h(x_s) \neq y_s\}$

Online learning - IID

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes
$\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\}$

IID $$(x_t, y_t) \sim \nu \quad y_t = h_*(x_t)$$

Corollary Under the conditions of the theorem (i.e., there exists an $h_* \in \mathcal{H}$ such that $R(h_*) = 0$, $(x_i, y_i) \overset{iid}{\sim} \nu$, and $\widehat{h} = \arg\min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{h(x_i) \neq y_i\}$) we have $\mathbb{E}[R(\widehat{h})] \leq \int_{\epsilon=0}^{d} \mathbb{P}(R(\widehat{h}) \geq \epsilon) \leq \frac{2\log(|\mathcal{H}|)}{n}$

Online learning - IID

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes $\sum_{t=1}^T \mathbf{1}\{h_t(x_t) \neq y_t\}$

IID $$(x_t, y_t) \sim \nu \quad y_t = h_*(x_t)$$

Corollary Under the conditions of the theorem (i.e., there exists an $h_* \in \mathcal{H}$ such that $R(h_*) = 0$, $(x_i, y_i) \overset{iid}{\sim} \nu$, and $\widehat{h} = \arg\min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{h(x_i) \neq y_i\}$) we have $\mathbb{E}[R(\widehat{h})] \leq \int_{\epsilon=0}^d \mathbb{P}(R(\widehat{h}) \geq \epsilon) \leq \frac{2\log(|\mathcal{H}|)}{n}$

$$\mathbb{E}\left[\sum_{t=1}^T \mathbf{1}\{h_t(x_t) \neq y_t\}\right] \leq 1 + \sum_{t=2}^T \mathbb{E}[\mathbb{P}(h_t(x_t) \neq y_t)]$$
$$\leq 1 + \sum_{t=2}^T \mathbb{E}[R(h_t)] \leq 1 + \sum_{t=2}^T \frac{2\log(|\mathcal{H}|)}{t-1} \leq 2 + 2\log(|\mathcal{H}|)\log(T)$$

# of mistakes grows only logarithmically!

Online learning - Adversarial

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes
$\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\}$

Adversarial (x_t, y_t) arbitrary $y_t = h_*(x_t)$

Online learning - Adversarial

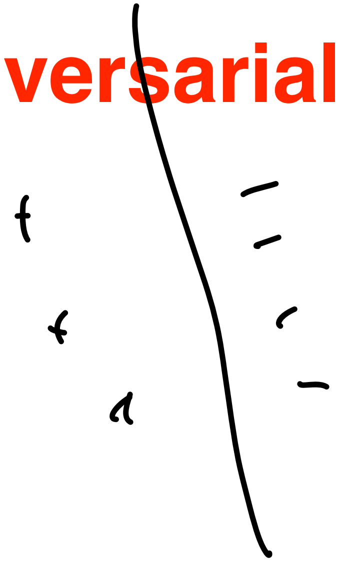
```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks $h_t \in \mathcal{H}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
```

Goal:
Minimize mistakes
$\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\}$

Adversarial (x_t, y_t) arbitrary $y_t = h_*(x_t)$

We know learning theory! Choose $h_t \in \arg\min_{h \in \mathcal{H}} \sum_{s=1}^{t-1} \mathbf{1}\{h(x_s) \neq y_s\}$?

Online learning - Adversarial



```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
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Claim There exists a sequence $\{(x_t, y_t)\}_{t=1}^{T}$ and $\widehat{h}_t \in \arg\min_{h \in \mathcal{H}} \sum_{s=1}^{t-1} \mathbf{1}\{h(x_s) \neq y_s\}$ such that the strategy makes $\min\{|\mathcal{H}|, T\}$ mistakes.

Hint: many classifiers achieve minimum, assume adversary knows your tie-breaking strategy

Online learning - Adversarial

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Halving Algorithm

```
Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$
Initialize: $V_1 = \mathcal{H}$
for $t = 1, 2, \ldots$
    $x_t$ arrives
    Player picks a $h_t \in V_t : \sum_{h \in V_t} \mathbf{1}\{h(x_t) = h_t(x_t)\} > \sum_{h \in V_t} \mathbf{1}\{h(x_t) = -h_t(x_t)\}$
    $y_t$ is revealed
    Player receives loss $\ell(h_t, (x_t, y_t)) = \mathbf{1}\{h_t(x_t) \neq y_t\}$
    Update $V_{t+1} = \{h \in V_t : h(x_t) = y_t\}$
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Online learning - Adversarial

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```

Either the algorithm doesn't make mistake, or *at least half* of hypotheses are discarded

Online learning - Adversarial

```
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Minimize mistakes
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Adversarial (x_t, y_t) arbitrary $y_t = h_*(x_t)$

Theorem: Fix a finite hypothesis class $\mathcal{H}$ so that $|\mathcal{H}| < \infty$ and for all $h \in \mathcal{H}$ we have $h(x) \in \{-1, 1\}$. Let $(x_1, y_1), \dots, (x_n, y_n)$ where x_t is arbitrary and $y_t = h_*(x_t)$ for some $h_* \in \mathcal{H}$. Then if h_t is recommended by the Halving algorithm, we have that $\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} \leq \log_2(|\mathcal{H}|)$

Online learning

Assuming that your data is IID is a **very** strong assumption that is almost never true in practice. Online learning is a different paradigm that makes no assumptions but still yields meaningful guarantees.

Assuming there exists a perfect classifier h_*:

- When x_t is drawn IID, empirical risk minimization results in only a number of mistakes that grows like $\log(T)\log(H)$

- When x_t is chosen adversarially empirical risk minimization can do arbitrarily badly. But there exist smarter approaches (like Halving algorithm) that make only $\log(H)$ mistakes

Questions?

Online learning in non-separable case

Online learning

Goal: Minimize regret wrt best

$$\max_{h \in \mathcal{H}} \sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} - \mathbf{1}\{h(x_t) \neq y_t\}$$

Input: $\mathcal{H}$ with $|\mathcal{H}| < \infty$

for $t = 1, 2, \ldots$

- x_t arrives
- Player picks $h_t \in \mathcal{H}$
- y_t is revealed
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Settings of interest:

IID	$(x_t, y_t) \sim \nu$
Adversarial	(x_t, y_t) arbitrary

Online learning

Goal: Minimize regret wrt best

$$\max_{h\in\mathcal{H}} \sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} - \mathbf{1}\{h(x_t) \neq y_t\}$$

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$\texttt{for}$ $t = 1, 2, \dots$
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Settings of interest:

IID $(x_t, y_t) \sim \nu$

Choose $h_t \in \arg\min_{h\in\mathcal{H}} \sum_{s=1}^{t-1} \mathbf{1}\{h(x_s) \neq y_s\}$

Corollary Under the conditions of the theorem (i.e., $(x_i, y_i) \overset{iid}{\sim} \nu$, and $\widehat{h} = \arg\min_{h\in\mathcal{H}} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{h(x_i) \neq y_i\}$) and $|\mathcal{H}| \geq n$, we have $\mathbb{E}[R(\widehat{h})] - R(h_*) \leq \sqrt{\frac{8\log(|\mathcal{H}|)}{n}}$

$$\implies \max_{h\in\mathcal{H}} \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} - \mathbf{1}\{h(x_t) \neq y_t\}\right] \leq \sqrt{8T\log(|\mathcal{H}|)}$$

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```

$h \longleftrightarrow i$, $\quad d = |\mathcal{H}|$

Settings of interest:

IID $\quad (x_t, y_t) \sim \nu$

$z_t(i) = \mathbb{1}\{h(x_t) \neq y_t\}$

Adversarial $\quad (x_t, y_t)$ arbitrary

Theorem: If $z_t \in [0,1]^d$ $\forall t$, and I_t, p_t are chosen by exponential weights then
$\max_{i\in[d]} \mathbb{E}\left[\sum_{t=1}^T \langle I_t, z_t\rangle - \langle \mathbf{e}_i, z_t\rangle\right] = \max_{i\in[d]} \sum_{t=1}^T \langle p_t, z_t\rangle - \langle \mathbf{e}_i, z_t\rangle \leq \sqrt{T\log(d)/2}$

$$\Longrightarrow \max_{h\in\mathcal{H}} \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} - \mathbf{1}\{h(x_t) \neq y_t\}\right] \leq \sqrt{T\log(|\mathcal{H}|)/2}$$

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Adversarial (x_t, y_t) arbitrary

$$\implies \max_{h\in\mathcal{H}} \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{1}\{h_t(x_t) \neq y_t\} - \mathbf{1}\{h(x_t) \neq y_t\}\right] \leq \sqrt{T\log(|\mathcal{H}|)/2}$$

Online learning

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Questions?

Exponential weights

Expert prediction

Suppose $b_t \in [0,1]^d$ is a vector of **d** experts predictions of tomorrow's temperature.

	t=1	*t=2*	*t=3*	*t=4*	*t=5*	…
Expert 1						
Expert 2						
Expert 3						

Expert prediction

Suppose $b_t \in [0,1]^d$ is a vector of **d** experts predictions of tomorrow's temperature.

	t=1	*t*=2	*t*=3	*t*=4	*t*=5	…
Expert 1						
Expert 2						
Expert 3						

$$z_t(i) = |b_t(i) - y_t|$$

*i*th expert's prediction

True temperature

```
Input: $d$ experts
for $t = 1, 2, \ldots$
    Player picks $p_t \in \Delta_d$ and plays $I_t \sim p_t$
    Adversary simultaneously reveals expert losses $z_t \in [0,1]^d$
    Player pays loss $\langle p_t, z_t \rangle = \mathbb{E}[z_t(I_t)]$
```

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Goal: Minimize regret wrt best

$$\max_{i \in [d]} \sum_{t=1}^{T} \langle p_t, z_t \rangle - \langle \mathbf{e}_i, z_t \rangle$$

Expert prediction

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```

Exponential weights algorithm

```
Input: $d$ experts, $\eta > 0$
Initialize: $w_1 \in [1, \ldots, 1]^\top \in \mathbb{R}^d$
for $t = 1, 2, \ldots$
    Player plays $I_t \sim p_t$ where $p_t(i) = w_t(i) / \sum_{j=1}^d w_t(j)$
    Adversary simultaneously reveals expert losses $z_t \in [0,1]^d$
    Player pays loss $\langle p_t, z_t \rangle = \mathbb{E}[z_t(I_t)]$
    Player updates weights $w_{t+1}(i) = w_t(i) \exp(-\eta z_t(i))$
```

Expert prediction

Goal: Minimize regret wrt best $$\max_{i \in [d]} \sum_{t=1}^{T} \langle p_t, z_t \rangle - \langle \mathbf{e}_i, z_t \rangle$$

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```

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$$\max_{i \in [d]} \mathbb{E}\left[\sum_{t=1}^{T} \langle I_t, z_t \rangle - \langle \mathbf{e}_i, z_t \rangle \right] = \max_{i \in [d]} \sum_{t=1}^{T} \langle p_t, z_t \rangle - \langle \mathbf{e}_i, z_t \rangle \leq \frac{\log(d)}{\eta} + \frac{T\eta}{8}$$

Choosing $\eta = \sqrt{\frac{8 \log(d)}{T}}$ gives regret bound of $\sqrt{T \log(d)/2}$

Expert prediction

Goal: Minimize regret wrt best

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Exponential weights algorithm, proof: Let $W_t = \sum_{i=1}^d w_t(i)$ so that

Expert prediction

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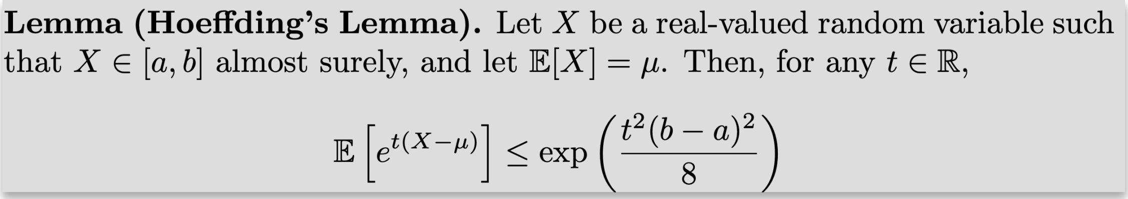
Exponential weights algorithm, proof: Let $W_t = \sum_{i=1}^d w_t(i)$ so that

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &= \sum_{t=1}^{T} \log \frac{W_{t+1}}{W_t} \\
&= \sum_{t=1}^{T} \log \left(\sum_{i=1}^{d} \frac{w_{t+1}(i)}{W_t} \right) \\
&= \sum_{t=1}^{T} \log \left(\sum_{i=1}^{d} \frac{w_t(i) \exp(-\eta z_t(i))}{W_t} \right) \\
&= \sum_{t=1}^{T} \log \left(\sum_{i=1}^{d} p_t(i) \exp(-\eta z_t(i)) \right) \\
&= \sum_{t=1}^{T} \log \left(\exp(-\eta \mathbb{E}[z_t(I_t)]) \sum_{i=1}^{d} p_t(i) \exp(-\eta (z_t(i) - \mathbb{E}[z_t(I_t)])) \right) \\
&= \sum_{t=1}^{T} -\eta \mathbb{E}[z_t(I_t)] + \log \left(\mathbb{E}[\exp(-\eta(z_t(I_t) - \mathbb{E}[z_t(I_t)]))] \right) \\
&\leq \sum_{t=1}^{T} -\eta \mathbb{E}[z_t(I_t)] + \eta^2/8
\end{aligned}$$

$$\begin{aligned}
\log \frac{W_{T+1}}{W_1} &\geq \log \frac{w_{T+1}(i)}{W_1} \\
&= -\log(d) + \log \left(\prod_{t=1}^{T} \exp(-\eta z_t(i)) \right) \\
&= -\log(d) - \sum_{t=1}^{T} \eta z_t(i)
\end{aligned}$$



Lemma (Hoeffding's Lemma). Let X be a real-valued random variable such that $X \in [a, b]$ almost surely, and let $\mathbb{E}[X] = \mu$. Then, for any $t \in \mathbb{R}$,

$$\mathbb{E}\left[e^{t(X-\mu)}\right] \leq \exp\left(\frac{t^2(b-a)^2}{8}\right)$$

$$\implies \sum_{t=1}^{T} \eta \mathbb{E}[z_t(I_t)] - \sum_{t=1}^{T} \eta z_t(i) \leq \log(d) + \eta^2 T/8$$