Stochastic M.A.B

2 coins $\rightarrow$ Heads or Tails

$\quad\quad\quad\quad\quad\quad\quad 1 \quad\quad 0$

$X_1 \; , \; X_2$

$\mu_1 \quad \mathbb{E}[X_1] \; , \; \mathbb{E}[X_2] \leftarrow \mu_2$

$\quad \frac{1}{2} \quad\quad\quad \frac{1}{2} + \varepsilon$

$\varepsilon > 0$

Coin 2 is the best option

If we choose coin 1, losing out by ε.

$I_t \leftarrow$ denote the coin index you choose

$$R(T) = T \cdot \left(\frac{1}{2} + \varepsilon\right) - \sum_{t=1}^{T} \mathbb{E}\left[\mu_{I_t}\right]$$

K - arms $\quad \mu_1 > \mu_2 \geq \cdots \geq \mu_K$

Sampling an arm

Pulling an arm $\quad X \sim \mathcal{D}_i \to$ dist of arm i

$$\mathbb{E}[X] = \mu_i$$

— x —

$X \sim \mathcal{D}_1 \qquad \mathbb{E}[X] = \mu_1$

$$X_1, \ldots, X_n \overset{i.i.d}{\sim} \mathcal{D}_1$$

$$\bar{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$|\mu_1 - \bar{\mu}_1|$ w.h.p $1 - \delta$

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Basics of Probability inequalities

Markov's inequality:

For any positive R.V X,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

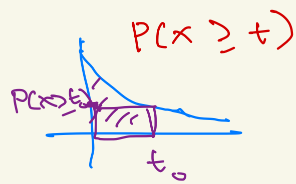
$[t > 0]$

Proof:

$$\mathbb{E}[X] = \int_{t=0}^{\infty} \mathbb{P}(X \geq t)\,dt$$

$$\geq t_0\,\mathbb{P}(X \geq t_0)$$

$$\mathbb{P}(X \geq t_0) \leq \frac{\mathbb{E}[X]}{t_0}$$



$$\mathbb{P}(|x| \geq t) \leq \frac{\mathbb{E}[|x|]}{t}$$

Chebychev's ineq $\quad \mathbb{P}(x^2 \geq t^2) \leq \frac{\mathbb{E}[x^2]}{t^2}$

Chernoff Bound :

Let $Z_1, \dots, Z_n$ be mean-zero IID RV and define

$$\psi_Z(\lambda) = \log\left(\mathbb{E}[\exp(\lambda Z_1)]\right)$$

Then $\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} Z_i \geq \varepsilon\right)$

$$\leq \inf_{\lambda > 0} \exp\left(-n\varepsilon\lambda + n\,\psi_Z(\lambda)\right)$$

A & B → Ind. R.V

$E[AB] = E[A] \cdot E[B]$

Proof: $\mathbb{P}\left(\sum_{i=1}^{n} z_i \geq \varepsilon n\right)$

$$= \mathbb{P}\left(\lambda \sum_{i=1}^{n} z_i \geq \lambda \varepsilon n\right)$$

$$= \mathbb{P}\left(\exp\left(\lambda \sum_{i=1}^{n} z_i\right) \geq \exp(\underline{\lambda \varepsilon} n)\right)$$

$$\leq e^{-\lambda \varepsilon n} \, \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{n} z_i\right)\right]$$

$$= e^{-\lambda \varepsilon n} \, \mathbb{E}\left[\prod_{i=1}^{n} \exp(\lambda z_i)\right]$$

$$= e^{-\lambda \varepsilon n} \prod_{i=1}^{n} \underbrace{\mathbb{E}\left[\exp(\lambda z_i)\right]}_{\mathbb{E}[\exp(\lambda z_1)]}$$

$$= e^{-\lambda \varepsilon n} \cdot \mathbb{E}\left[\exp(\lambda z_1)\right]^{n}$$

$$= \exp\left(-\lambda \varepsilon n + n \underbrace{\log \mathbb{E}\left[\exp(\lambda z_1)\right]}_{\psi_z(\lambda)}\right)$$

$$\psi_Z(\lambda) = \log\left(\mathbb{E}\left[\exp(\lambda Z)\right]\right)$$

Ex:

$$Z \sim \mathcal{N}(0, \sigma^2)$$

$$\mathbb{E}\left[e^{\lambda Z}\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\lambda z} e^{-z^2/2\sigma^2} \, dz$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(z-\lambda\sigma^2)^2/2\sigma^2} \cdot e^{\lambda^2\sigma^2/2} \, dz$$

$$= e^{\lambda^2\sigma^2/2} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(z-\lambda\sigma^2)^2/2\sigma^2} \, dz}_{1}$$

$$= e^{\lambda^2\sigma^2/2}$$

$$\psi_Z(\lambda) = \lambda^2\sigma^2/2$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i \geq \varepsilon\right) \leq \inf_{\lambda > 0} \exp\left(-n\varepsilon\lambda + n \overbrace{\lambda^2\sigma^2/2}^{\psi_Z(\lambda)}\right)$$

$$\lambda = \varepsilon/\sigma^2$$

$$= \exp\left(-\frac{n\varepsilon^2}{2\sigma^2}\right)$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} z_i \geq \varepsilon\right) \leq \exp\Bigg(\underbrace{-\frac{n\varepsilon^2}{2\sigma^2}}_{\delta}\Bigg)$$

$$z \sim N(0, 1)$$

$$\exp\left(+\frac{n\varepsilon^2}{2\sigma^2}\right) = 1/\delta$$

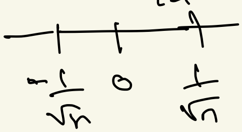
$$\frac{n\varepsilon^2}{2\sigma^2} = \log(1/\delta)$$

$$\varepsilon = \sigma\sqrt{\frac{2\log(1/\delta)}{n}}$$

w.p $1-\delta$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} z_i < \sqrt{\frac{2\log(1/\delta)}{n}}\right) \leftarrow$$

w.p $1-\delta \rightarrow$ $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} z_i > -\sqrt{\frac{2\log(1/\delta)}{n}}\right)$



$X_1, \dots, X_n \overset{i.i.d}{\sim} \mathcal{D}_1$ s.t $\mathbb{E}[X_i] = \mu_1$ $\rightarrow N(\mu_1, 1)$

$$\bar{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_1$$

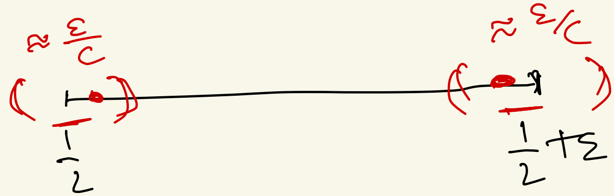
$$|\bar{\mu}_1 - \mu_1|$$

$Z_i = X_i - \mu_1$ $\qquad$ $\mathbb{E}[Z_i] = 0$

$Z_1, \dots, Z_n$ - i.i.d

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_1) < \sqrt{\frac{2\log(1/\delta)}{n}}\right) \geq 1-\delta$$

$$\iff \mathbb{P}\left(\bar{\mu}_1 - \mu_1 < \sqrt{2\log(1/\delta)/n}\right) \geq 1-\delta$$



$\frac{c}{\varepsilon^2}$

— x —

Sub-gaussian distributions

Defn: We say a mean-zero R.V Z is σ^2-sub-gaussian if $\underbrace{\log \mathbb{E}[e^{\lambda Z}]}_{\psi_Z(\lambda)} \le \frac{\sigma^2 \lambda^2}{2}$

Lemma: Hoeffding's ineq

Let X be a mean zero R.V and $X \in [a,b]$ almost surely. Then $\log(\mathbb{E}[e^{\lambda X}]) \le \frac{(b-a)^2}{8}\lambda^2$ for all λ.

$\Leftrightarrow$ X is $\frac{(b-a)^2}{4}$- sub-gaussian

— x —

$Y_i \sim$ Bernoulli (θ) $\quad Y_i \in \{0,1\}$

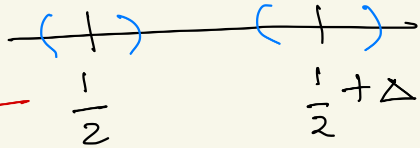
w.p $1-\delta$

$$\frac{1}{n}\sum_{i}^{n} Y_i \le \theta + \sqrt{\frac{\log(1/\delta)}{2n}}$$

and w.p $1-\delta$

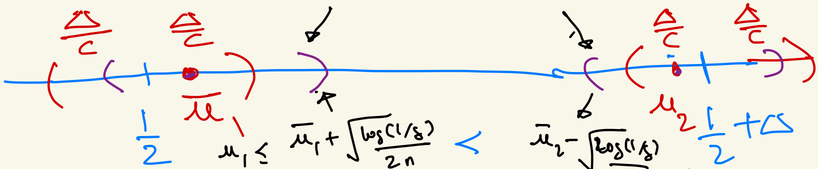
$$\frac{1}{n}\sum_{i=1}^{n} Y_i \ge \theta - \sqrt{\frac{\log(1/\delta)}{2n}}$$

$\frac{C}{\Delta^2}$



I_t ← index you choose in round t

$$R(T) = T \cdot \left(\frac{1}{2}+\Delta\right) - \sum_{t=1}^{T} \mathbb{E}\left[\mu_{I_t}\right]$$



$$\mu_1 \le \bar{\mu}_1 + \sqrt{\frac{\log(1/\delta)}{2n}} < \bar{\mu}_2 - \sqrt{\frac{\log(1/\delta)}{2m}} \le \mu_2$$

$$R(T) = \min\left\{ \frac{100}{\Delta^2}\cdot\Delta + 0, \frac{\Delta T}{2} \right\}$$

$$\boxed{\delta = \frac{1}{T^2}}$$

$$\min\left\{ \frac{100 \log T}{\Delta}, \frac{\Delta T}{2} \right\} \le C \cdot \sqrt{T \log T}$$

— x —

$$R(T) \le \min\left\{ \sum_{i=2}^{K} \frac{\log T}{\Delta_i}, \sqrt{KT\log T} \right\}$$

$$\Delta_i = \mu_1 - \mu_i$$