Lin-UCB

for $t = 1, 2, \ldots$

Learner chooses $x_t \in \mathcal{X}$

Nature reveals $y_t = \langle x_t, \theta_* \rangle + \eta_t$ $\qquad$ η_t 1-sub-Gaussian

Goal: Minimize $\quad R_T = \max_{x \in \mathcal{X}} \sum_{t=1}^{T} \langle x - x_t, \theta_* \rangle$

$$= \sum_{t=1}^{T} \langle x_* - x_t, \theta_* \rangle$$

Let $C_t \subset \mathbb{R}^d$ s.t. $\theta_* \in C_t \ \forall t.$ w.p. $\geq 1 - \delta$

$$UCB_t(x) = \operatorname*{argmax}_{\theta \in C_t} \langle x, \theta \rangle$$

$$x_t = \operatorname*{argmax}_{x \in \mathcal{X}} UCB_t(x)$$

For MAB $\quad C_t = \prod_{i=1}^{d} \left[\hat{\mu}_i - \sqrt{\frac{c \log(t)}{T_i(t)}}, \hat{\mu}_i + \sqrt{\frac{c \log(t)}{T_i(t)}} \right]$

$$\hat{\theta}_t = \operatorname*{argmin}_{\theta} \sum_{s=1}^{t} (y_s - \langle x_s, \theta \rangle)^2 + \lambda \|\theta\|_2^2$$

$$= \underbrace{\left(\lambda I + \sum_{s=1}^{t} x_s x_s^T \right)^{-1}}_{V_t} \underbrace{\sum_{s=1}^{t} x_s y_s}_{S_t}, \quad \hat{\theta}_t = V_t^{-1} S_t$$

$$V_0 = \lambda I \qquad \|x\|_A^2 = x^T A x$$

$C_t = \{ \theta : \|\theta - \hat{\theta}_t\|_{V_t}^2 \leq \beta_t \}$ for some deterministic sequence β_t

Thm] If $\max_{x \in \mathcal{X}} \|x\| \le L$, $\max_{x \in \mathcal{X}} |\langle x, \theta_* \rangle| \le 1$, then

$$R_T \le \sqrt{8 T \beta_T \log\left(\frac{|V_T|}{|V_0|}\right)} \le \sqrt{8 d T \beta_T \log\left(\frac{d\lambda + TL^2}{d\lambda}\right)}.$$

$$\langle \theta_*, x_* \rangle \le UCB_t(x_*) \le UCB_t(x_t) =: \langle \tilde{\theta}_t, x_t \rangle$$

$$\tilde{\theta}_t := \arg\max_{\theta \in C_t} \langle \theta, x_t \rangle$$

$$\begin{aligned}
\langle \theta_*, x_* - x_t \rangle &\le \langle \tilde{\theta}_t, x_t \rangle - \langle \theta_*, x_t \rangle \\
&= \langle \tilde{\theta}_t - \theta_*, x_t \rangle \\
&= \langle V_t^{1/2} (\tilde{\theta}_t - \theta_*), V_t^{-1/2} x_t \rangle \\
&\le \| V_t^{1/2} (\tilde{\theta}_t - \theta_*) \|_2 \cdot \| V_t^{-1/2} x_t \|_2 \\
&= \|\tilde{\theta}_t - \theta_*\|_{V_t} \cdot \|x_t\|_{V_t^{-1}} \\
&\le \left(\|\tilde{\theta}_t - \hat{\theta}_t\|_{V_t} + \|\hat{\theta}_t - \theta_*\|_{V_t} \right) \|x_t\|_{V_t^{-1}} \\
&\le 2\sqrt{\beta_t}\, \|x_t\|_{V_t^{-1}}
\end{aligned}$$

Cauchy-Schwartz
$\langle x, y \rangle \le \|x\|_2 \|y\|_2$

Assume $\beta_t \ge 1$

$$|\langle \theta_*, x_* - x_t \rangle| \le 2$$

$$\langle \theta_*, x_* - x_t \rangle \le 2 \wedge 2\sqrt{\beta_t}\, \|x_t\|_{V_t^{-1}} \le 2\sqrt{\beta_t} \left(1 \wedge \|x_t\|_{V_t^{-1}}\right)$$

$$R_T = \sum_{t=1}^{T} \langle \theta_*, x_* - x_t \rangle$$

$$\leq \sqrt{T \sum_{t=1}^{T} \langle \theta_*, x_* - x_t \rangle^2}$$

$$\leq 2\sqrt{T \beta_T \sum_{t=1}^{T} \left(1 \wedge \|x_t\|^2_{V_t^{-1}}\right)} .$$

For MAB $\quad x_t = e_{I_t}$

$$V_t = \mathrm{diag}([T_1(t), \ldots, T_d(t)]) + \cancel{I}$$

$$\|x_t\|^2_{V_t^{-1}} = \frac{1}{T_{I_t}(t)}$$

$$\sum_{t=1}^{T} \|x_t\|^2_{V_t^{-1}} = \sum_{i=1}^{d} \sum_{t : I_t = i} \|x_t\|^2_{V_t^{-1}}$$

$$= \sum_{i=1}^{d} \sum_{l=1}^{T_i(T)} \frac{1}{l}$$

$$\leq \sum_{i=1}^{d} \log(T_i(T))$$

$$\leq d \log(T)$$

Elliptic Potential Lemma

Fix V_0 and let $x_t \in \mathbb{R}^d$ be arbitrary sequence w/ $\|x_t\|_2 \le L$ and set $V_t = V_0 + \sum_{s=1}^{t} x_s x_s^T$

$$\sum_{t=1}^{T} \left(1 \wedge \|x_t\|^2_{V_t^{-1}}\right) \le 2 \log\left(\frac{|V_T|}{|V_0|}\right) \le 2d \log\left(\frac{T_r(V_0)+TL^2}{d|V_0|^{1/d}}\right) \boxed{}$$

__Proof__

$$V_t = V_{t-1} + x_t x_t^T$$

$$= V_{t-1}^{1/2}\left(I + V_{t-1}^{-1/2} x_t x_t^T V_{t-1}^{-1/2}\right) V_{t-1}^{1/2}$$

$$|V_t| = |V_{t-1}| \cdot \left| I + V_{t-1}^{-1/2} x_t x_t^T V_{t-1}^{-1/2} \right|$$

$$= |V_{t-1}| \cdot \left(1 + \|x_t\|^2_{V_{t-1}^{-1}}\right)$$

$$= |V_0| \prod_{s=1}^{t} \left(1 + \|x_s\|^2_{V_{s-1}^{-1}}\right)$$

$$\log \frac{|V_T|}{|V_0|} = \sum_{t=1}^{T} \log\left(1 + \|x_t\|^2_{V_t^{-1}}\right)$$

$$\sum_t \left(1 \wedge \|x_t\|^2_{V_t^{-1}}\right) \le \sum_t 2 \log\left(1 + \|x_t\|^2_{V_t^{-1}}\right) = 2 \log \frac{|V_T|}{|V_0|}$$

$$\leq \|x\|^2_{V_{t-1}^{-1}}$$

$$|V_T| = \prod_{i=1}^{d} \lambda_i \leq \left(\frac{1}{d} \sum_{i=1}^{d} \lambda_i \right)^d = \left(\frac{1}{d} \mathrm{Tr}(V_T) \right)^d$$

AMGM