Moment Generating Function. For any $\lambda \in \mathbb{R}$

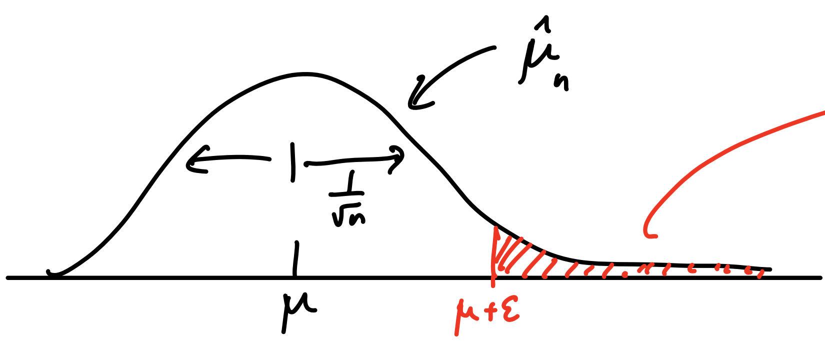
$$\varphi_X(\lambda) = \mathbb{E}[\exp(\lambda X)].$$

$$\frac{\partial^k}{\partial \lambda^k} \varphi_X(\lambda) \Big|_{\lambda=0} = \mathbb{E}[X^k]$$

$$e^{\lambda y} = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

Def) A mean-zero R.V. X is σ^2-sub-Gaussian if $\mathbb{E}[e^{\lambda X}] \le e^{\sigma^2 \lambda^2 / 2}$

Lemma) If $X_1, \dots, X_n$ are mean-zero IID σ^2-sub-Gaussian then $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon\right) \le e^{-n\varepsilon^2/2}$.



$\hat{\mu}_n = \frac{1}{n}\sum Y_i$

$Y_1, \dots, Y_n$

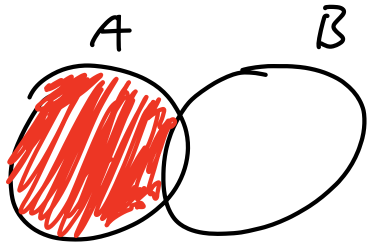
$\mathbb{E}[Y_i] = \mu$

$(Y_i - \mu)$ are sub-Gaussian

__Lemma__ | Hoeffding If X is mean zero R.V. w/ $X \in [a,b]$ then $\mathbb{E}[\exp(\lambda X)] \le \exp\left(\frac{(b-a)^2 \lambda^2}{8}\right)$

$$2\varepsilon_{\ell+1} = 2 \cdot 2^{-(\ell+1)} = 2^{-\ell} = \varepsilon_\ell$$

Input: n arms $\mathcal{X} = \{1, \ldots, n\}$, confidence level $\delta \in (0,1)$.
Let $\widehat{\mathcal{X}}_1 \leftarrow \mathcal{X}, \ell \leftarrow 1$
while $|\mathcal{X}_\ell| > 1$ **do**
  $\epsilon_\ell = 2^{-\ell}$
  Pull each arm in $\mathcal{X}_\ell$ exactly $\tau_\ell = \lceil 2\epsilon_\ell^{-2} \log(\frac{4\ell^2|\mathcal{X}|}{\delta}) \rceil$ times
  Compute the empirical mean of these rewards $\widehat{\theta}_{i,\ell}$ for all $i \in \mathcal{X}_\ell$
 $\mathcal{X}_{\ell+1} \leftarrow \mathcal{X}_\ell \setminus \{i \in \mathcal{X}_\ell : \max_{j \in \mathcal{X}_\ell} \widehat{\theta}_{j,\ell} - \widehat{\theta}_{i,\ell} > 2\epsilon_\ell\}$
 $\ell \leftarrow \ell + 1$
Output: $\widehat{\mathcal{X}}_{\ell+1}$ (or play the last arm forever in the regret setting)



$A \setminus B$

$$\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$$

$$\mathbb{P}(A \cup \underbrace{B \cup C}_{:=D}) \le \mathbb{P}(A) + \mathbb{P}(D)$$
$$\le \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$$

$$\mathcal{E}_{i,\ell} = \{ |\hat{\theta}_{i,\ell} - \theta_i| \le \varepsilon_\ell \}. \quad \text{Want } \mathcal{E} = \bigcap_{i=1}^{n} \bigcap_{\ell=1}^{\infty} \mathcal{E}_{i,\ell}$$

__Fact__ | For any events $A_1, \ldots, A_m$ $\mathbb{P}(\bigcup_{i=1}^m A_i) \le \sum_{i=1}^m \mathbb{P}(A_i)$. __Union Bound.__

$$\mathbb{P}(\mathcal{E}^c) = \mathbb{P}\left(\bigcup_{i=1}^{n} \bigcup_{\ell=1}^{\infty} \mathcal{E}_{i,\ell}^c\right)$$
$$\le \sum_{i=1}^{n} \sum_{\ell=1}^{\infty} \mathbb{P}(\mathcal{E}_{i,\ell}^c) = \sum_{i=1}^{n} \sum_{\ell=1}^{\infty} \frac{\delta}{4\ell^2 n} 2 = \frac{\delta}{2} \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \le \delta.$$

Assume $X_{i,t}$ is $\mathbb{E}[X_{i,t}] = \theta_i$, $(X_{i,t} - \theta_i)$ - $\overset{=1}{\sigma^2}$-sub-Gaussian

$$\mathbb{P}(\mathcal{E}_{i,\ell}^c) = \mathbb{P}(|\hat{\theta}_{i,\ell} - \theta_i| > \varepsilon_\ell)$$
$$= \mathbb{P}(\{\hat{\theta}_{i,\ell} - \theta_i > \varepsilon_\ell\} \cup \{\theta_i - \hat{\theta}_{i,\ell} > \varepsilon_\ell\})$$

$$= \underbrace{\mathbb{P}\left(\hat{\theta}_{i,\ell} - \theta_i > \varepsilon_\ell\right)}_{\circledast} + \mathbb{P}\left(\theta_i - \hat{\theta}_{i,\ell} > \varepsilon_\ell\right)$$

$$\circledast = \mathbb{P}\left(\frac{1}{\tau_\ell} \sum_{t=1}^{\tau_\ell} (X_{i,t} - \theta_i) > \varepsilon_\ell\right)$$

$-\log(x) = \log(\frac{1}{x})$

$\exp(\log(y)) = y$

$$\leq \exp\left(- \tau_\ell \varepsilon_\ell^2 / 2\right) \qquad \text{(Chernoff Bound)}$$

$$\leq \exp\left(- \log\left(\frac{4\ell^2 |\mathcal{X}|}{\delta}\right)\right)$$

$$= \frac{\delta}{4\ell^2 |\mathcal{X}|} = \boxed{\frac{\delta}{4\ell^2 n}}$$

Assume $\theta_1 > \theta_2 \geq \dots \geq \theta_n$

$\Delta_j = \theta_1 - \theta_j \quad \forall j. \implies \Delta_j \geq 0$

Show $1 \in \mathcal{X}_\ell \quad \forall \ell$ w.p. $\geq 1 - \delta$.

arm 1 is kicked out if $\exists j \neq 1$ s.t. $\hat{\theta}_{j,\ell} > \hat{\theta}_{1,\ell} + 2\varepsilon_\ell$

$$\hat{\theta}_{j,\ell} - \hat{\theta}_{1,\ell} = \left(\hat{\theta}_{j,\ell} - \theta_j\right) - \left(\hat{\theta}_{1,\ell} - \theta_1\right) - \Delta_j$$

$$\leq 2\varepsilon_\ell - \Delta_j$$

$$\leq 2\varepsilon_\ell. \implies 1 \in \mathcal{X}_\ell \quad \forall \ell$$

Suppose $\boxed{\Delta_i > 4\varepsilon_\ell.}$ Then

$$\max_{j\in\mathcal{X}_\ell}\hat{\theta}_{j,\ell} - \hat{\theta}_{i,\ell} \ge \hat{\theta}_{1,\ell} - \hat{\theta}_{i,\ell}$$

$$= (\hat{\theta}_{1,\ell}-\theta_1)-(\hat{\theta}_{i,\ell}-\theta_i) + \Delta_i$$

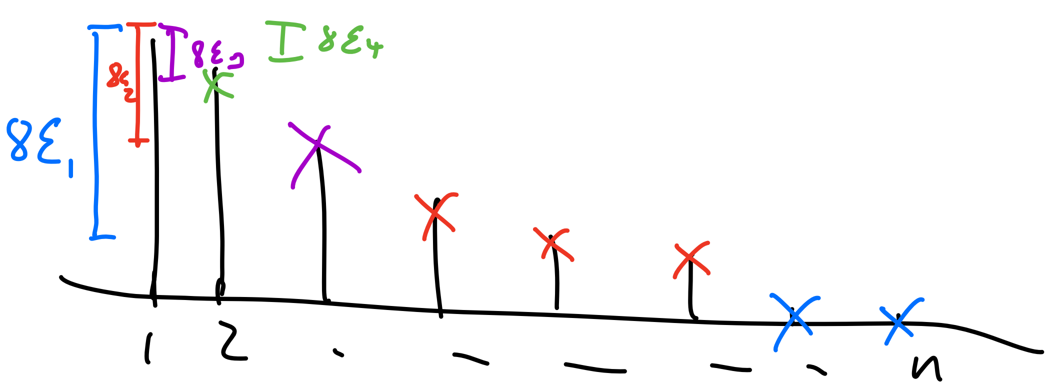
$$\ge -2\varepsilon_\ell + \Delta_i$$

$$> 2\varepsilon_\ell.$$

$$\implies i \notin \mathcal{X}_{\ell+1}$$

$$\equiv \quad \Delta_i > 8\varepsilon_{\ell+1}$$

$$\implies \max_{i\in\mathcal{X}_\ell}\Delta_i \le 8\varepsilon_\ell.$$



Lemma On $\mathcal{E}$ we have $1 \in \mathcal{X}_\ell \ \forall \ell$ and

$$\max_{i \in \mathcal{X}_\ell} \Delta_i \le 8 \varepsilon_\ell .$$

Question: How long until $\mathcal{X}_\ell = \{1\}$?

If $\Delta := \min_{i \neq 1} \Delta_i$ then $\mathcal{X}_\ell = \{1\}$ for $\ell > \log_2\left(\frac{8}{\Delta}\right)$.

$$\sum_{\ell=1}^{\log(8/\Delta)} |\mathcal{X}_\ell| \, \tau_\ell$$

$$= \sum_{\ell=1}^{\log(8/\Delta)} \sum_{i=1}^{n} \mathbb{1}\{i \in \mathcal{X}_\ell\} \tau_\ell$$

$$\le 2 \sum_{\ell=1}^{\log(8/\Delta)} \sum_{i=2}^{n} \mathbb{1}\{i \in \mathcal{X}_\ell\} \tau_\ell$$

$$= 2 \sum_{i=2}^{n} \boxed{\sum_{\ell=1}^{\log(8/\Delta)} \mathbb{1}\{i \in \mathcal{X}_\ell\} \tau_\ell} = T_i$$

$$\le 2 \sum_{i=2}^{n} \sum_{\ell=1}^{\log(8/\Delta)} \mathbb{1}\{\Delta_i \le 8 \varepsilon_\ell\} \tau_\ell$$

$$= 2 \sum_{i=2}^{n} \sum_{\ell=1}^{\log(8/\Delta_i)} \tau_\ell$$

$$= 2 \varepsilon_\ell^{-2} \log\left(\frac{4 \ell^2 n}{\delta}\right)$$

$$\leq \sum_{i=2}^{n} 4 \log\left(\frac{4\log^2(8/\Delta_i)n}{\delta}\right) \underbrace{\sum_{\ell=1}^{\log(8/\Delta_i)} \underbrace{\varepsilon_\ell^{-2}}_{=4^\ell}}$$

$$\leq c\, 4^{\log_2(8/\Delta_i)}$$

$$= c\, 2^{\log_2((8/\Delta_i)^2)}$$

$$= c \left(\frac{8}{\Delta_i}\right)^2$$

$$\leq c' \sum_{i=2}^{n} \Delta_i^{-2} \log\left(\frac{\log(\Delta_i^{-2}) \cdot n}{\delta}\right)$$

$$\sum_{k=0}^{m} a^k = \frac{1-a^{m+1}}{1-a} \Rightarrow \sum_{k=1}^{m} a^k = -1 + \frac{1-a^{m+1}}{1-a}$$

$$= \frac{a^{m+1}-1-a+1}{a-1}$$

$$\leq \frac{a^{m+1}}{a-1}$$

W.p. $\geq 1-\delta$, for all i

We showed above that $T_i \leq \Delta_i^{-2} \log\left(\frac{\log(\Delta_i^{-2})\, n}{\delta}\right)$

What is non-adaptive rate?

$$\Delta = \min_i \Delta_i$$

Non-adaptive $\quad n \Delta^{-2} \log\left(\frac{\log(\Delta^{-2}) \cdot n}{\delta} \right)$

At each time t, algorithm picks $I_t \in [n]$

and gets reward $X_{I_t, t}$ where $\mathbb{E}[X_{i,t}] = \theta_i$

and $(X_{i,t} - \theta_i)$ is sub-Gaussian.

$$\text{Regret} \quad R_T := \max_{i \in [n]} \mathbb{E}\left[\sum_{t=1}^{T} X_{i,t} - X_{I_t,t} \right]$$

$$= \max_{i \in [n]} \mathbb{E}\left[\sum_{t=1}^{T} \theta_i - \theta_{I_t} \right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} T_i \quad \Delta_i \right]$$

$$\Delta_i = \max_j \theta_j - \theta_i \qquad = \sum_{i=1}^{n} \mathbb{E}[T_i] \Delta_i$$

Assume $\Delta_i \leq 1$

$$R_T = \mathbb{E}\left[\sum_{i=1}^{n} T_i \Delta_i\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} T_i \Delta_i \mathbb{1}\{\mathcal{E}\} + \sum_{i=1}^{n} T_i \Delta_i \mathbb{1}\{\mathcal{E}^c\}\right]$$

$$\leq \sum_{i=2}^{n} \Delta_i^{-1} \log\left(\frac{\log(\Delta_i^{-2}) \cdot n}{\delta}\right) \cancel{\mathbb{P}(\mathcal{E})}^{\leq 1}$$

$$+ T \underbrace{\mathbb{P}(\mathcal{E}^c)}_{\leq \delta}$$

If $\delta = 1/T$ and $T >> n$ then

$$R_T \leq \sum_i \Delta_i^{-1} \log\left(\log(\Delta_i^{-1}) \cdot T\right) + 1$$

$$R_T = \;= \sum_{\ell=1}^{\log(8/\Delta)} \sum_{i=1}^{n} \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathfrak{I}_\ell \Delta_i$$

$$\leq 2 \sum_{\ell=1}^{\log(8/\Delta)} \sum_{i=2}^{n} \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathfrak{I}_\ell \Delta_i$$

$$= 2 \sum_{i=2}^{n} \boxed{\sum_{\ell=1}^{\log(8/\Delta)} \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathfrak{I}_\ell \Delta_i} = T_i$$

Fix ν.

$$\leq 2 \sum_{i=2}^{n} \sum_{\ell=1}^{\log(8/\Delta)} \mathbb{1}\{\Delta_i \leq 8 \varepsilon_\ell\} \mathfrak{I}_\ell \Delta_i$$

$$= 2 \sum_{i=2}^{n} \sum_{\ell=1}^{\log(8/\Delta_i)} \underbrace{\mathfrak{I}_\ell}_{= 2\varepsilon_\ell^{-2} \log\left(\frac{4\ell^2 n}{\delta}\right)} \Delta_i$$

$$= \underbrace{\sum_{i:\Delta_i \leq \nu} \sum_{\ell=1}^{\log(8/\Delta_i)} \mathfrak{I}_\ell \Delta_i}_{\leq \nu T} + \sum_{\underline{i:\Delta_i > \nu}} \sum_{\ell=1}^{\log(8/\Delta_i)} \mathfrak{I}_\ell \Delta_i$$

$$= \sum_{i:\Delta_i > \nu} \sum_{\ell=1}^{\log_2(8/\max\{\nu, \Delta_i\})} \mathfrak{I}_\ell \Delta_i$$

$$= \sum_{i:\Delta_i > \nu} (\Delta_i \vee \nu)^{-1} \log\left(\frac{\log((\Delta_i \vee \nu)^{-2}) \cdot n}{\delta}\right)$$

On $\mathcal{E}$:

$$R_T \le \min_{\gamma}\left(\gamma T + \sum_{i:\Delta_i > \gamma} (\Delta_i \vee \gamma)^{-1} \log\left(\frac{\log((\Delta_i \vee \gamma)^{-2}) \cdot n}{\delta}\right)\right)$$

$$\le \sqrt{nT \log\left(\frac{n \cdot \log(T)}{\delta}\right)}$$

$\delta = 1/T$

$$\mathbb{E}[R_T] = \mathbb{E}\left[R_T \mathbb{1}\{\mathcal{E}\} + R_T \mathbb{1}\{\mathcal{E}^c\}\right]$$

$$\le \min\left\{\sqrt{nT\log(T)}, \sum_{i:\Delta_i>0} \Delta_i^{-1} \log\left(\log(\Delta_i^{-2})T\right)\right\}$$

Theorem) (Lai and Robbins 1985) Suppose for any $i: \Delta_i > 0$

$\mathbb{E}[T_i] \le T^{\alpha}$ $\forall$ $\alpha > 0$. Then

$$\lim_{T\to\infty} \frac{R_T}{\log(T)} \to 2 \sum_{i:\Delta_i>0} \Delta_i^{-1}$$