

Moment Generating Function. For any  $\lambda \in \mathbb{R}$

$$\varphi_X(\lambda) = \mathbb{E}[\exp(\lambda X)].$$

$$\left. \frac{\partial^k}{\partial \lambda^k} \varphi_X(\lambda) \right|_{\lambda=0} = \mathbb{E}[X^k]$$

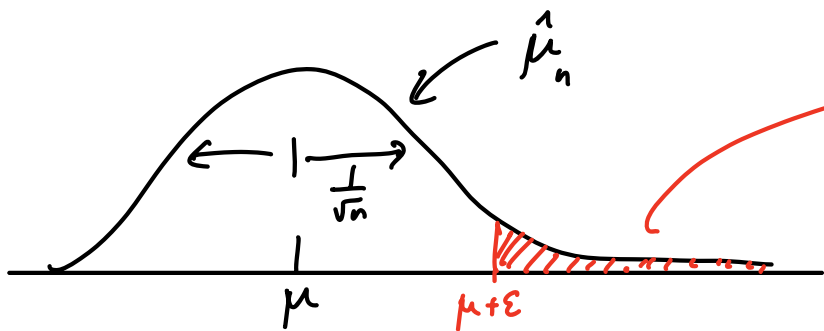
$$e^{\lambda y} = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

Def) A mean-zero R.V.  $X$  is  $\sigma^2$ -sub-Gaussian

if  $\mathbb{E}[e^{\lambda X}] \leq e^{\sigma^2 \lambda^2 / 2}$

Lemma) If  $X_1, \dots, X_n$  are mean-zero IID

$\sigma^2$ -sub-Gaussian then  $P\left(\frac{1}{n} \sum_{i=1}^n X_i > \varepsilon\right) \leq \underline{\underline{e^{-n\varepsilon^2/2}}}$ .



$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n Y_i$$

$$Y_1, \dots, Y_n$$

$$\mathbb{E}[Y_i] = \mu$$

$(Y_i - \mu)$  are sub-Gaussian

Lemma 9 Hoeffding If  $X$  is mean zero R.V.

w/  $X \in [a, b]$  then  $\mathbb{E}[\exp(\lambda X)] \leq \exp\left(\frac{(b-a)^2 \lambda^2}{8}\right)$

$$2\epsilon_{\ell+1} = 2 \cdot 2^{-(\ell+1)} = 2^{-\ell} = \epsilon_{\ell}$$

**Input:**  $n$  arms  $\mathcal{X} = \{1, \dots, n\}$ , confidence level  $\delta \in (0, 1)$ .

Let  $\hat{\mathcal{X}}_1 \leftarrow \mathcal{X}, \ell \leftarrow 1$

**while**  $|\mathcal{X}_{\ell}| > 1$  **do**

$$\epsilon_{\ell} = 2^{-\ell}$$

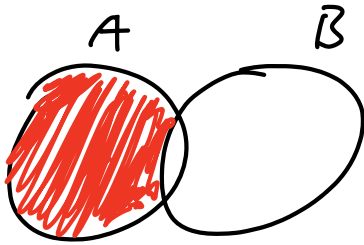
Pull each arm in  $\mathcal{X}_{\ell}$  exactly  $\tau_{\ell} = \lceil 2\epsilon_{\ell}^{-2} \log(\frac{4\ell^2 |\mathcal{X}|}{\delta}) \rceil$  times

Compute the empirical mean of these rewards  $\hat{\theta}_{i,\ell}$  for all  $i \in \mathcal{X}_{\ell}$

$$\mathcal{X}_{\ell+1} \leftarrow \mathcal{X}_{\ell} \setminus \{i \in \mathcal{X}_{\ell} : \max_{j \in \mathcal{X}_{\ell}} \hat{\theta}_{j,\ell} - \hat{\theta}_{i,\ell} > 2\epsilon_{\ell}\}$$

$$\ell \leftarrow \ell + 1$$

**Output:**  $\hat{\mathcal{X}}_{\ell+1}$  (or play the last arm forever in the regret setting)



$A \setminus B$

$$P(A \cup B) \leq P(A) + P(B)$$

$$P(A \cup \underbrace{B \cup C}_{:=D}) \leq P(A) + P(D) \leq P(A) + P(B) + P(C)$$

$$\mathcal{E}_{i,\ell} = \{|\hat{\theta}_{i,\ell} - \theta_i| \leq \epsilon_{\ell}\}. \quad \text{Want } \mathcal{E} = \bigcap_{i=1}^n \bigcap_{\ell=1}^{\infty} \mathcal{E}_{i,\ell}$$

Fact For any events  $A_1, \dots, A_m$   $P(\bigcup_{i=1}^m A_i) \leq \sum_{i=1}^m P(A_i)$ . Union Bound.

$$P(\mathcal{E}^c) = P\left(\bigcup_{i=1}^n \bigcup_{\ell=1}^{\infty} \mathcal{E}_{i,\ell}^c\right) \leq \sum_{i=1}^n \sum_{\ell=1}^{\infty} P(\mathcal{E}_{i,\ell}^c) = \sum_{i=1}^n \sum_{\ell=1}^{\infty} \frac{\delta}{4\ell^2 n} = \frac{\delta}{2} \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \leq \delta.$$

Assume  $X_{i,t}$  is  $\mathbb{E}[X_{i,t}] = \theta_i$ ,  $(X_{i,t} - \theta_i) \stackrel{!}{\sim} \sigma^2$ -sub-Gaussian.

$$P(\mathcal{E}_{i,\ell}^c) = P(|\hat{\theta}_{i,\ell} - \theta_i| > \epsilon_{\ell}) = P(\{\hat{\theta}_{i,\ell} - \theta_i > \epsilon_{\ell}\} \cup \{\theta_i - \hat{\theta}_{i,\ell} > \epsilon_{\ell}\})$$

$$= \underbrace{\mathbb{P}(\hat{\theta}_{i,\ell} - \theta_i > \varepsilon_\ell)}_{(*)} + \mathbb{P}(\theta_i - \hat{\theta}_{i,\ell} > \varepsilon_\ell)$$

$$(*) = \mathbb{P}\left(\frac{1}{\varepsilon_\ell} \sum_{t=1}^{\varepsilon_\ell} (X_{i,t} - \theta_i) > \varepsilon_\ell\right)$$

$$-\log(x) = \log\left(\frac{1}{x}\right)$$

$$\exp(\log(y)) = y$$

$$\leq \exp\left(-\varepsilon_\ell \varepsilon_\ell^2 / 2\right)$$

(Chernoff Bound)

$$\leq \exp\left(-\log\left(\frac{4\varepsilon_\ell^2 |\mathcal{X}|}{\delta}\right)\right)$$

$$= \frac{\delta}{4\varepsilon_\ell^2 |\mathcal{X}|} = \boxed{\frac{\delta}{4\varepsilon_\ell^2 n}}$$

Assume  $\theta_1 > \theta_2 \geq \dots \geq \theta_n$

$$\Delta_j = \theta_1 - \theta_j \quad \forall j. \quad \Rightarrow \quad \Delta_j \geq 0$$

Show  $1 \in \mathcal{X}_\ell \quad \forall \ell \quad \text{w.p.} \geq 1 - \delta.$

arm 1 is kicked out if  $\exists j \neq 1$  s.t.  $\hat{\theta}_{j,\ell} > \hat{\theta}_{1,\ell} + 2\varepsilon_\ell$

$$\hat{\theta}_{j,\ell} - \hat{\theta}_{1,\ell} = (\hat{\theta}_{j,\ell} - \theta_j) - (\hat{\theta}_{1,\ell} - \theta_1) - \Delta_j$$

$$\leq 2\varepsilon_\ell - \Delta_j$$

$$\leq 2\varepsilon_\ell. \quad \Rightarrow \quad 1 \in \mathcal{X}_\ell \quad \forall \ell$$

Suppose

$$\Delta_i > 4\epsilon_\ell.$$

Then

$$\max_{j \in \mathcal{X}_\ell} \hat{\theta}_{j,\ell} - \hat{\theta}_{i,\ell} \geq \hat{\theta}_{1,\ell} - \hat{\theta}_{i,\ell}$$

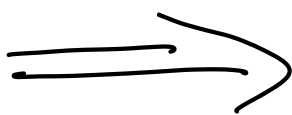
$$= (\hat{\theta}_{1,\ell} - \theta_1) - (\hat{\theta}_{i,\ell} - \theta_i) + \Delta_i$$

$$\geq -2\epsilon_\ell + \Delta_i$$

$$> 2\epsilon_\ell.$$

$$\implies i \notin \mathcal{X}_{\ell+1}$$

$$\equiv \Delta_i > 8\epsilon_{\ell+1}$$



$$\max_{i \in \mathcal{X}_\ell} \Delta_i \leq 8\epsilon_\ell.$$



Lemma On  $\mathcal{E}$  we have  $1 \in \mathcal{X}_\ell \forall \ell$  and

$$\max_{i \in \mathcal{X}_\ell} \Delta_i \leq \delta \varepsilon_\ell.$$

Question: How long until  $\mathcal{X}_\ell = \{1\}$ ?

If  $\Delta := \min_{i \neq 1} \Delta_i$  then  $\mathcal{X}_\ell = \{1\}$  for  $\ell > \log_2\left(\frac{\delta}{\Delta}\right)$ .

$$\begin{aligned} & \sum_{\ell=1}^{\log(\delta/4)} |\mathcal{X}_\ell| \mathcal{I}_\ell \\ &= \sum_{\ell=1}^{\log(\delta/\Delta)} \sum_{i=1}^n \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathcal{I}_\ell \\ &\leq 2 \sum_{\ell=1}^{\log(\delta/\Delta)} \sum_{i=2}^n \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathcal{I}_\ell \\ &= 2 \sum_{i=2}^n \sum_{\ell=1}^{\log(\delta/\Delta)} \mathbb{1}\{i \in \mathcal{X}_\ell\} \mathcal{I}_\ell \\ &\leq 2 \sum_{i=2}^n \sum_{\ell=1}^{\log(\delta/\Delta)} \mathbb{1}\{\Delta_i \leq \delta \varepsilon_\ell\} \mathcal{I}_\ell \\ &= 2 \sum_{i=2}^n \sum_{\ell=1}^{\log(\delta/\Delta_i)} \mathcal{I}_\ell \\ & \quad \underline{\hspace{1cm}} \\ &= 2 \varepsilon_\ell^{-2} \log\left(\frac{4\ell^2 n}{\delta}\right) \end{aligned}$$

$$\leq \sum_{i=2}^n 4 \log \left( \frac{4 \log^2(8/\Delta_i) n}{\delta} \right) \sum_{\ell=1}^{\log(8/\Delta_i)} \underbrace{\varepsilon_\ell}_{=4^\ell}^{-2}$$

$$\leq c 4^{\log_2(8/\Delta_i)}$$

$$= c 2^{\log_2(8/\Delta_i)^2}$$

$$= c \left( \frac{8}{\Delta_i} \right)^2$$

$$\leq c' \sum_{i=2}^n \Delta_i^{-2} \log \left( \frac{\log(\Delta_i^{-2}) \cdot n}{\delta} \right)$$

$$\sum_{k=0}^m a^k = \frac{1-a^{m+1}}{1-a} \Rightarrow \sum_{k=1}^m a^k = -1 + \frac{1-a^{m+1}}{1-a}$$

$$= \frac{a^{m+1} - 1 - a + 1}{a-1}$$

$$\leq \frac{a^{m+1}}{a-1}$$