Lemma Chernoff-Bound

Let $Z_1, \dots, Z_n$ be mean-zero IID R.V. and define

$$\psi_Z(\lambda) = \log\left(\mathbb{E}[\exp(\lambda Z_1)]\right).$$

Then $$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i > \varepsilon\right) \leq \inf_{\lambda} \exp(-n\varepsilon\lambda + n\psi_Z(\lambda)).$$

$$\frac{d}{d\lambda}\mathbb{E}[e^{\lambda Z}]\Big|_{\lambda=0} = \mathbb{E}[Z e^{\lambda Z}]\Big|_{\lambda=0} = \mathbb{E}[Z]$$

$$\frac{d^2}{\partial\lambda^2}\mathbb{E}[e^{\lambda Z}]\Big|_{\lambda=0} = \mathbb{E}[Z^2 e^{\lambda Z}]\Big|_{\lambda=0} = \mathbb{E}[Z^2]$$

Ex.

$Z \sim \mathcal{N}(0, \sigma^2)$

$$\mathbb{E}[e^{\lambda Z}] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{\lambda z} e^{-z^2/2\sigma^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-(z-\lambda\sigma^2)^2/2\sigma^2}\, e^{\lambda^2\sigma^2/2}\, dz$$

$$= e^{\lambda^2\sigma^2/2} \cdot 1$$

$$\Rightarrow \psi_Z(\lambda) = \lambda^2\sigma^2/2.$$

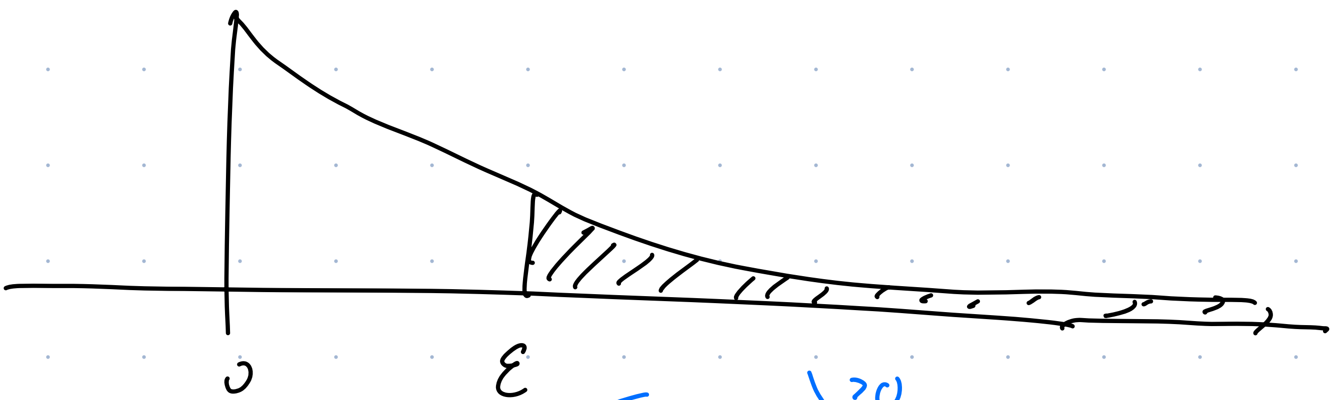
$$\Rightarrow \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i > \varepsilon\right) \le \inf_{\lambda} \exp\left(-n\varepsilon\lambda + n\lambda^2\sigma^2/2\right) \quad (*)$$

$$\frac{d}{d\lambda}\left(-n\varepsilon\lambda + n\lambda^2\sigma^2/2\right) = -n\varepsilon + n\lambda\sigma^2$$

$$\Rightarrow \lambda = \varepsilon/\sigma^2$$

$$(*) = \exp\left(-n\varepsilon^2/2\sigma^2\right)$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i > \varepsilon\right) \le \exp\left(-\frac{n\varepsilon^2}{2\sigma^2}\right)$$



For any $\lambda \ge 0$

Proof

$$\mathbb{P}\left(\sum_{i=1}^{n} Z_i > \varepsilon n\right) = \mathbb{P}\left(\lambda \cdot \sum_{i=1}^{n} Z_i > \varepsilon\lambda n\right)$$

$$= \mathbb{P}\left(\exp\left(n\sum_{i=1}^{n} Z_i\right) > \exp(\varepsilon\lambda n)\right)$$

If A, B are indep R.V.

$\mathbb{E}[AB] = \mathbb{E}[A]\,\mathbb{E}[B]$

$$\leq e^{-\lambda \varepsilon n}\, \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{n} z_i\right)\right]$$

$$= e^{-\lambda \varepsilon n}\, \mathbb{E}\left[\prod_{i=1}^{n} \exp(\lambda z_i)\right]$$

$$= e^{-\lambda \varepsilon n} \prod_{i=1}^{n} \underbrace{\mathbb{E}\left[e^{\lambda z_i}\right]}_{= \mathbb{E}[e^{\lambda z_1}]}$$

$$= e^{\lambda \varepsilon n} \ldots$$

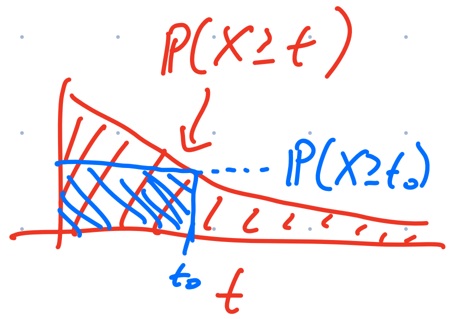
$$= e^{-\lambda \varepsilon n}\, \mathbb{E}\left[e^{\lambda z_1}\right]^n$$

$$= \exp\left(-\lambda \varepsilon n + n\, \varphi_z(\lambda)\right)$$

Lemma Markov's inequality. For positive R.V.

X, $\quad \mathbb{P}(X > t) \leq \dfrac{\mathbb{E}[X]}{t}$.

Pf. $\quad \mathbb{E}[X] = \displaystyle\int_{t=0}^{\infty} \mathbb{P}(X \geq t)\, dt$



$$t_0\, \mathbb{P}(X \geq t_0) \leq \mathbb{E}[X].$$

$$\mathbb{P}(|X| \geq t) \leq \frac{\mathbb{E}[|X|]}{t}$$

Chebychev's inequality $\quad \mathbb{P}(X^2 > t^2) \leq \dfrac{\mathbb{E}[X^2]}{t^2}$

Def We say a mean-zero R.V. Z is <u>σ^2- sub-Gaussian</u> if $\log \mathbb{E}[e^{\lambda Z}] \leq \sigma^2 \lambda^2 / 2$.

$$f_Z(x) \propto \frac{1}{(x+a)^3} \qquad \mathbb{E}[e^{\lambda Z}] \simeq \int_0^{\infty} \frac{1}{(x+a)^3} e^{\lambda x}\, dx$$

Lemma Hoeffding's inequality.

Let X be a mean-zero R.V. w/ $X \in [a,b]$ almost surely. Then $\log(\mathbb{E}[e^{\lambda X}]) \le \frac{(b-a)^2 \lambda^2}{8}$.

$$Y_i \sim \text{Bernoulli}(\theta). \quad Y_i \in \{0,1\}$$

$$Y_i - \theta \in [-\theta, 1-\theta] \Rightarrow (Y_i - \theta) \text{ is } \tfrac{1}{4}\text{-sub-Gauss}$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \theta) > \varepsilon\right) \le e^{-2n\varepsilon^2}$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i > \theta + \varepsilon\right) \le e^{-2n\varepsilon^2}$$

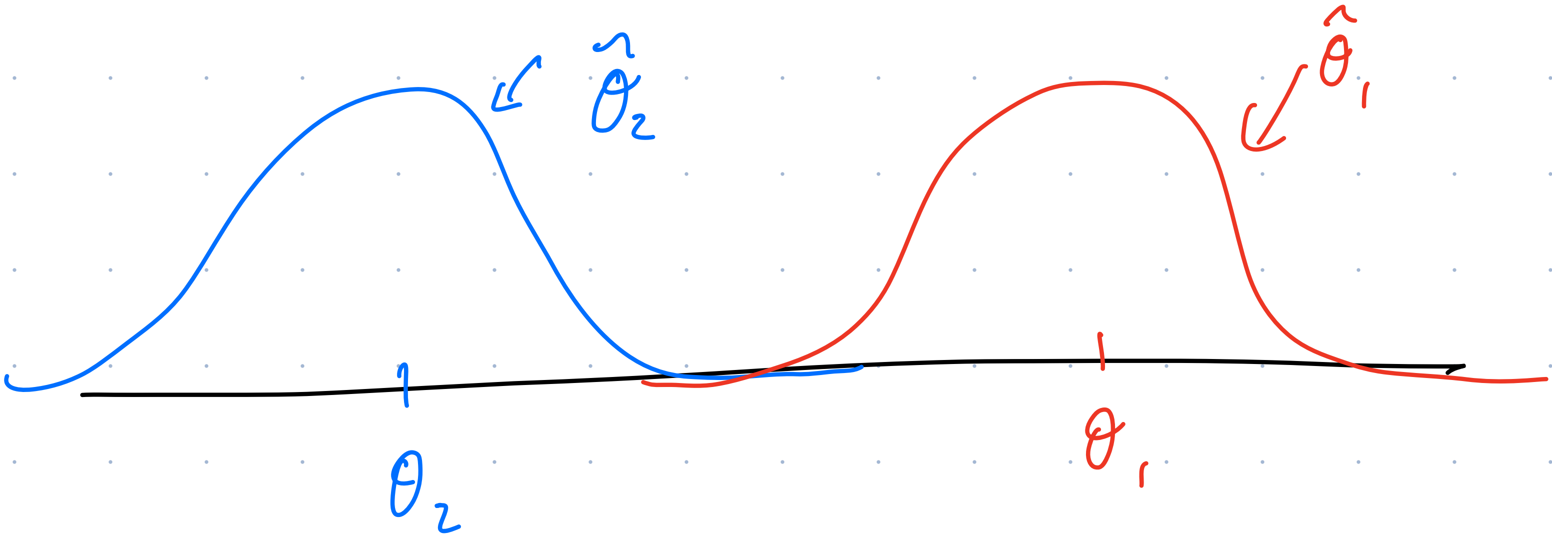
Fix $\delta \in (0,1)$ and set $= \delta$. Solve for ε.

$$\varepsilon = \sqrt{\frac{\log(1/\delta)}{2n}}.$$

With probability at least $1-\delta$ we have

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \le \theta + \sqrt{\frac{\log(1/\delta)}{2n}}.$$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i < \theta - \varepsilon\right) \le e^{-2n\varepsilon^2}$$



W.p. $\geq 1-\delta$ we have

$$\left\{ \hat{\theta}_2 \le \theta_2 + \sqrt{\frac{\log(1/\delta)}{2q}} \right\} =: \mathcal{E}_2$$

W.p. $\geq 1-\delta$ we have

$$\left\{ \hat{\theta}_1 \ge \theta_1 - \sqrt{\frac{\log(1/\delta)}{2q}} \right\} =: \mathcal{E}_1$$

$$\mathbb{P}(\mathcal{E}_i) \ge 1-\delta.$$

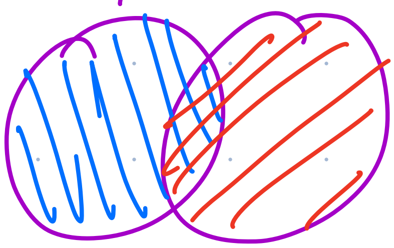
$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) = 1 - \mathbb{P}((\mathcal{E}_1 \cap \mathcal{E}_2)^c)$$

$$= 1 - \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c)$$

$$\geq 1 - \mathbb{P}(\mathcal{E}_1^c) - \mathbb{P}(\mathcal{E}_2^c)$$

$$\geq 1 - 2\delta.$$

$\mathcal{E}_1$ $\mathcal{E}_2$



$\mathbb{P}(\mathcal{E}_1^c)$ $\mathbb{P}(\mathcal{E}_2^c)$

$$\mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c) = \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c) - \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_2^c)$$

$$\leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c)$$

w.p. $\geq 1 - 2\delta$

$$\hat{\theta}_1 \geq \theta_1 - \sqrt{\frac{\log(1/\delta)}{2\,3}}$$

$$\hat{\theta}_2 \leq \theta_2 + \sqrt{\frac{\log(1/\delta)}{2\,3}}$$

$$\Rightarrow \quad \hat{\theta}_1 > \hat{\theta}_2 \quad \text{if} \quad \theta_1 - \theta_2 \geq 2\sqrt{\frac{\log(1/\delta)}{2S}}$$

$$\Delta := \theta_1 - \theta_2 \qquad \text{or} \qquad S \geq \frac{2\log(1/\delta)}{\Delta^2}$$

$$R_T \leq \Delta T$$

$$R_T = \sum_{i=1}^{n} \Delta_i \mathbb{E}[T_i]$$

$$= \Delta \mathbb{E}[T_2]$$

$$= \Delta \mathbb{E}\left[T_2 \mathbb{1}\{\mathcal{E}_1 \cap \mathcal{E}_2\} + T_2 \mathbb{1}\{\mathcal{E}_1^c \cup \mathcal{E}_2^c\}\right]$$

$$\leq \Delta \mathbb{E}\left[S \mathbb{1}\{\mathcal{E}_1 \cap \mathcal{E}_2\} + T \mathbb{1}\{\mathcal{E}_1^c \cup \mathcal{E}_2^c\}\right]$$

$$= \Delta S \, \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) + \Delta T \, \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c)$$

$$\leq \Delta S + \Delta T \cdot 2\delta.$$

$$\leq \frac{2\log(2T)}{\Delta} + \Delta.$$

Choose $\delta = \frac{1}{2T} \Rightarrow \beta = \frac{2\log(2T)}{\Delta^2}$)

$$R_T \le \min\left\{ \Delta T, \frac{2\log(2T)}{\Delta} + \Delta \right\}$$

$$\le \sqrt{2T\log(2T)}$$