Problem set 4 — Solutions

CSE 534 Autumn 2026

Problem 1 (Partial trace).

1. Let $\{|i\rangle\}$ be an orthonormal basis for $\mathcal{H}_B$ and $\{|j\rangle\}$ be an orthonormal basis for $\mathcal{H}_C$. On the previous homework you proved that $\{|i\rangle \otimes |j\rangle\}$ is an orthonormal basis for $\mathcal{H}_B \otimes \mathcal{H}_C$. Notice also that

$$(\mathbb{I}_A \otimes \langle i|_B \otimes \mathbb{I}_C)(\mathbb{I}_A \otimes \mathbb{I}_B \otimes \langle j|_C) = (\mathbb{I}_A \otimes \mathbb{I}_B \otimes \langle j|_C)(\mathbb{I}_A \otimes \langle i|_B \otimes \mathbb{I}_C) = (\mathbb{I}_A \otimes \langle i|_B \otimes \langle j|_C)$$

   Making use of linearity and the definition of the partial trace, the result follows.

2. From the definition of the trace

$$\begin{aligned}\operatorname{tr}(\rho_A) &= \operatorname{tr}\left(\sum_i (\mathbb{I}_A \otimes \langle i|_B)\rho_{AB}(\mathbb{I}_A \otimes |i\rangle_B)\right) = \sum_i \operatorname{tr}\big((\mathbb{I}_A \otimes |i\rangle_B)(\mathbb{I}_A \otimes \langle i|_B)\rho_{AB}\big) \\ &= \operatorname{tr}\left(\left(\mathbb{I}_A \otimes \sum_B |i\rangle\langle i|\right)\rho_{AB}\right) = \operatorname{tr}(\rho_{AB}) = 1\end{aligned}$$

   We now argue that $\rho_A \succeq 0$. Let $|\psi\rangle \in \mathcal{H}_A$. Then

$$\langle\psi|\rho|\psi\rangle = \sum_i \langle\psi|\,(\mathbb{I}_A \otimes \langle i|_B)\rho_{AB}(\mathbb{I}_A \otimes |i\rangle_B)\,|\psi\rangle = \sum_i \underbrace{\langle\psi|\,\langle i|_B\,\rho_{AB}\,|i\rangle_B\,|\psi\rangle}_{\geq 0}$$

   which is the sum of positive terms since ρ_{AB} is positive.

3. This follows because the measurement operator M_i from any POVM on A commutes with the action of the partial trace.

Problem 2.

1. Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ be a decomposition. Let $q = \operatorname{tr}(\Pi\rho)$ and for each i, let $q_i = \langle\psi_i|\Pi|\psi_i\rangle$. From the decomposition, we have that $q = \sum_i p_i q_i$.

   Next, for each i, define $|v_i\rangle \stackrel{\text{def}}{=} \Pi\,|\psi_i\rangle \,/\, \sqrt{q}$. These are not normalized vectors, but by linearity

$$\rho' = \frac{\Pi\rho\Pi}{q} = \sum_i p_i\,|v_i\rangle\langle v_i|\,.$$

Then, using the triangle inequality, the hint, and Cauchy-Schwarz (in order)

$$\begin{aligned}\left\|\rho-\rho'\right\|_1 &\leq \sum_i p_i \left\| |\psi_i\rangle\langle\psi_i| - |v_i\rangle\langle v_i| \right\|_1 \\ &= \sum_i p_i \left\| |\psi_i\rangle + |v_i\rangle \right\| \cdot \left\| |\psi_i\rangle - |v_i\rangle \right\| \\ &\leq \sqrt{\left(\sum_i p_i \left\| |\psi_i\rangle + |v_i\rangle \right\|^2\right)\left(\sum_i p_i \left\| |\psi_i\rangle - |v_i\rangle \right\|^2\right)}.\end{aligned}$$

Next, observe that we can calculate that

$$\begin{aligned}\sum_i p_i \left\| |\psi_i\rangle + |v_i\rangle \right\|^2 &= \sum_i p_i \left(\left\| |\psi\rangle_i \right\|^2 + \left\| |v_i\rangle \right\|^2 + 2\langle v_i|\psi_i\rangle \right) \\ &= \sum_i p_i \left(1 + \frac{q_i}{q} + \frac{2q_i}{\sqrt{q}}\right) \\ &= 1 + \left(\frac{1}{q} + \frac{2}{\sqrt{q}}\right)\sum_i p_i q_i \\ &= 2 + 2\sqrt{q}.\end{aligned}$$

Similarly, we can verify that $\sum_i p_i \left\| |\psi_i\rangle - |v_i\rangle \right\|^2 \leq 2 - 2\sqrt{q}$. Returning to the bound, we get that

$$\left\|\rho-\rho'\right\|_1 \leq \sqrt{(2+2\sqrt{q})(2-2\sqrt{q})} = 2\sqrt{1-q} = 2\sqrt{\epsilon}.$$

2. Consider the state $|\psi\rangle\sqrt{1-\epsilon}\,|0\rangle + \sqrt{\epsilon}\,|1\rangle$. If $\Pi = |0\rangle\langle 0|$, then $\mathrm{tr}(\Pi|\psi\rangle\langle\psi|) = 1-\epsilon$, and the post measurement state is $|0\rangle$. Then,

$$\left\| |\psi\rangle\langle\psi| - |0\rangle\langle 0| \right\|_1 = \mathrm{tr}\sqrt{\begin{pmatrix} -\epsilon & \sqrt{\epsilon\sqrt{1-\epsilon}} \\ \sqrt{\epsilon\sqrt{1-\epsilon}} & \epsilon \end{pmatrix}^2} = \sqrt{\epsilon \cdot \mathbb{I}_2} = 2\sqrt{\epsilon}.$$

Problem 3. From the last problem set, we proved that the 1-norm of a difference of density matrices is the maximum distinguishing probability over all distinguish measurements. Therefore, the left-hand side is the maximum distinguishing probability restricted to acting on A while the right-hand side is the maximum distinguishing probability allowed to act on A and B. As one is a subset of the other, the inequality holds. (This is known as the data processing inequality).

Problem 4 (Simultaneous change of basis)**.**

1. (a) Since $A^2 = B^2 = \mathbb{I}$, the eigenvalues are $\in \{\pm 1\}$. However, neither matrix can have both eigenvalues as $+1$ or -1 as then the matrix would equal $\mathbb{I}$ or $-\mathbb{I}$ and then commute with the other.

(b) Let $|v\rangle$ be an eigenvector of B with $B|v\rangle = \lambda|v\rangle$. Then,

$$B(A|v\rangle) = -AB|v\rangle = -\lambda A|v\rangle.$$

Therefore, $A|v\rangle$ is the other eigenvector, as the eigenvalues are distinct. So A maps the eigenvectors of B to each other, and likewise B maps the eigenvectors of A to each other.

(c) If $B = |v_0\rangle\langle v_0| - |v_1\rangle\langle v_1|$ let $U = |0\rangle|v_0\rangle + |1\rangle|v_1\rangle$. Then,

$$UBU^\dagger = |0\rangle\langle 0| - |1\rangle\langle 1| = Z.$$

Since A swaps the eigenvectors of B, $UAU^\dagger = |0\rangle\langle 1| + |1\rangle\langle 0| = X$.

2. Anticommutation implies that each pair $A_iB_i = -B_iA_i$. Identify a unitary U_i using part 1 per pair. Then, the block diagonal matrix of U_i suffices as V.

3. (a) Assume $|v\rangle$ is an eigenvector of $(A+B)$ of eigenvalue λ. Note that the eigenvalue is non-zero, as A and B do not share eigenvectors, since they anticommute. Then

$$A|v\rangle = \frac{1}{\lambda}A(A+B)|v\rangle = \frac{1}{\lambda}(|v\rangle + AB|v\rangle) \in \text{span}(|v\rangle, AB|v\rangle),$$
$$B|v\rangle = \frac{1}{\lambda}B(A+B)|v\rangle = \frac{1}{\lambda}(|v\rangle + BA|v\rangle) = \frac{1}{\lambda}(|v\rangle - AB|v\rangle) \in \text{span}(|v\rangle, AB|v\rangle).$$

So, $A|v\rangle$ and $B|v\rangle$ are in the span as well. Since $A(AB)|v\rangle = B|v\rangle$ and $B(AB)|v\rangle = -A|v\rangle$, it follows that both A and B preserve the vector space spanned by $|v\rangle$ and $AB|v\rangle$.

(b) Iteratively, identify an eigenvector $|v\rangle$ of $A+B$ and use it to decompose the vector space into blocks by defining a block as the span of $|v\rangle$ and $AB|v\rangle$. Each block has dimension at most 2. Since d is finite, this eventually halts. By the previous, both A and B are block diagonal with this decomposition.

(c) A one-dimensional block cannot exist, as the corresponding blocks would not anticommute.