Problem set 3 — Solutions

CSE 534 Autumn 2026

Problem 1 (Indistinguishable states).

1. Write $|\psi\rangle = a|0\rangle + b|1\rangle$ and choose the orthogonal state $|\psi'\rangle = -b^*|0\rangle + a^*|1\rangle$. Expanding the two tensor products gives

$$|\psi\rangle \otimes |\psi^*\rangle = |a|^2|00\rangle + ab^*|01\rangle + ba^*|10\rangle + |b|^2|11\rangle,$$
$$|\psi'\rangle \otimes |\psi'^*\rangle = |b|^2|00\rangle - b^*a|01\rangle - ab^*|10\rangle + |a|^2|11\rangle.$$

   The cross terms cancel, and since $|a|^2 + |b|^2 = 1$, we obtain

$$|\psi\rangle \otimes |\psi^*\rangle + |\psi'\rangle \otimes |\psi'^*\rangle = |00\rangle + |11\rangle.$$

   Dividing both sides by $\sqrt{2}$ gives the desired Bell state.

2. The two states define the same density matrix, as a phase of $e^{i\theta}$ and a phase of $e^{-i\theta}$ both appear.

3. Both ensembles have density matrix $\mathbb{I}/2$, since

$$\frac{1}{2}(|0\rangle\langle 0|\,0 + |1\rangle\langle 1|\,1) = \frac{\mathbb{I}}{2} = \frac{1}{2}(|+\rangle\langle +|\,+ + |-\rangle\langle -|\,-).$$

Problem 2 (Expectation of an operator).

1. If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the expected outcome is $|\alpha|^2 - |\beta|^2 = \langle\psi|Z|\psi\rangle$.

2. Measure in the Hadamard basis, assigning $+1$ to $|+\rangle$ and -1 to $|-\rangle$. The expectation is $\langle\psi|X|\psi\rangle$ because $X = |+\rangle\langle +| + -|-\rangle\langle -|$.

3. $\langle Z\rangle_\rho = \operatorname{tr}(Z\rho)$.

Problem 3 (Partial trace).

1. Fix orthonormal bases $\{|b\rangle\}$ and $\{|c\rangle\}$ for $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. By the definition of partial trace, tracing out C first gives

$$\operatorname{tr}_C(\rho_{ABC}) = \sum_c (\mathbb{I}_{AB} \otimes \langle c|)\rho_{ABC}(\mathbb{I}_{AB} \otimes |c\rangle).$$

Tracing out B from this expression, we obtain

$$\begin{aligned}\mathrm{tr}_B(\mathrm{tr}_C(\rho_{ABC})) &= \sum_b (\mathbb{I}_A \otimes \langle b|)\,\mathrm{tr}_C(\rho_{ABC})(\mathbb{I}_A \otimes |b\rangle) \\ &= \sum_{b,c} (\mathbb{I}_A \otimes \langle b| \otimes \langle c|)\rho_{ABC}(\mathbb{I}_A \otimes |b\rangle \otimes |c\rangle) \\ &= \mathrm{tr}_{BC}(\rho_{ABC}).\end{aligned}$$

The last equality follows because $\{|b\rangle \otimes |c\rangle\}_{b,c}$ is an orthonormal basis for $\mathcal{H}_B \otimes \mathcal{H}_C$. Reversing the order of the two sums gives exactly the same expression, so $\mathrm{tr}_C(\mathrm{tr}_B(\rho_{ABC})) = \mathrm{tr}_{BC}(\rho_{ABC})$ as well.

2. Let $\sigma_A = \mathrm{tr}_B(\sigma_{AB})$. We show that σ_A is positive semidefinite and has trace 1.

   For any $|v\rangle \in \mathcal{H}_A$, the definition of partial trace gives

$$\begin{aligned}\langle v|\,\sigma_A\,|v\rangle &= \sum_b \langle v|\,(\mathbb{I}_A \otimes \langle b|)\sigma_{AB}(\mathbb{I}_A \otimes |b\rangle)\,|v\rangle \\ &= \sum_b \langle v, b|\,\sigma_{AB}\,|v, b\rangle \\ &\geq 0.\end{aligned}$$

   The last inequality holds because σ_{AB} is positive semidefinite, so each summand is nonnegative. Thus σ_A is positive semidefinite and hence Hermitian.

   Finally, let $\{|a\rangle\}$ be an orthonormal basis for $\mathcal{H}_A$. We calculate

$$\begin{aligned}\mathrm{tr}(\sigma_A) &= \sum_a \langle a|\,\sigma_A\,|a\rangle \\ &= \sum_{a,b} \langle a, b|\,\sigma_{AB}\,|a, b\rangle \\ &= \mathrm{tr}(\sigma_{AB}) = 1.\end{aligned}$$

   Therefore, σ_A is a density matrix.

3. Let P_y project onto outcome y on the chosen qubit of A (and act as identity on the others). By the defining property of partial trace,

$$\underset{\sigma_A}{\mathbf{Pr}}(y) = \mathrm{tr}\big(P_y \sigma_A\big) = \mathrm{tr}\big((P_y \otimes \mathbb{I}_B)\sigma_{AB}\big) = \underset{\sigma_{AB}}{\mathbf{Pr}}(y).$$

Problem 4 (Density matrices of the W state)**.**

1. Observe that $|W_1^{(n)}\rangle = \frac{1}{\sqrt{n}}|1\rangle \otimes |0^{n-1}\rangle + \sqrt{\frac{n-1}{n}}\,|0\rangle \otimes |W_1^{(n-1)}\rangle$. Since $|0^n\rangle$ is orthogonal to $|W_1^{(n-1)}\rangle$ (different supports), then the reduced density matrix on the first qubit will have no cross terms and

so,

$$\rho_1 = \frac{n-1}{n} |0\rangle\langle 0| + \frac{1}{n} |1\rangle\langle 1| .$$

2. For the first two qubits, $|W_3\rangle = (|10\rangle + |01\rangle)|0\rangle / \sqrt{3} + |00\rangle |1\rangle / \sqrt{3}$. Hence, with $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$,

$$\rho_{12} = \frac{1}{3} |00\rangle\langle 00| + \frac{2}{3} |\Psi^+\rangle\langle\Psi^+| .$$

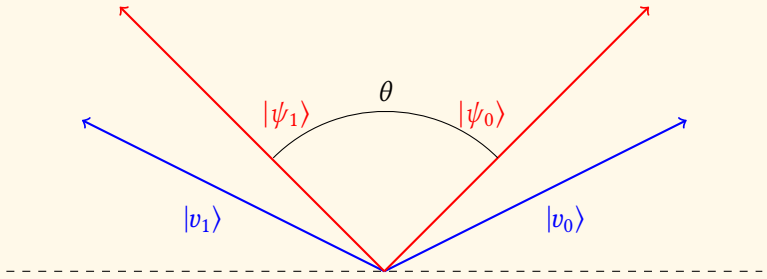
By symmetry, the same holds for every pair.

Problem 5 (Distinguishing quantum states)**.**

1. Let $P = |\psi_0\rangle\langle\psi_0|$. The strategy outputs 0 with probabilities $\mathrm{tr}(P\rho_0) = 1$ and $\mathrm{tr}(P\rho_1) = |\langle\psi_0|\psi_1\rangle|^2$. Hence

$$\Delta = 1 - |\langle\psi_0|\psi_1\rangle|^2.$$

2. Let $|v_0\rangle, |v_1\rangle$ be states in the span of $|\psi_0\rangle, |\psi_1\rangle$ such that $\langle v_0|v_1\rangle = 0$ and they are symmetric with respect to the angle bisector of $|\psi_0\rangle$ and $|\psi_1\rangle$ where $|v_i\rangle$ is closer to $|\psi_i\rangle$. On outcome $|v_i\rangle$ we guess $|\psi_i\rangle$. Illustratively, we have the following diagram.



It now remains to compute Δ for this strategy. Making use of the fact that the basis in which the measurement is made is orthonormal and the vectors $|v_0\rangle, |v_1\rangle$ are completely real, one has

$$\begin{aligned}
\Delta &= \Big|\mathbf{Pr}[P(|\psi\rangle_0) \text{ outputs } 0] - \mathbf{Pr}[P(|\psi\rangle_1) \text{ outputs } 0]\Big| \\
&= \Big|\big|\langle\psi_0|v_0\rangle\big|^2 - \big|\langle\psi_1|v_0\rangle\big|^2\Big| = \Big|\big|\|\psi_0\| \cdot \|v_0\| \cdot \cos(\angle(\psi_0, v_0))\big|^2 - \big|\|\psi_1\| \cdot \|v_0\| \cdot \cos(\angle(\psi_1, v_0))\big|^2\Big| \\
&= \Big|\cos^2(\angle(\psi_0, v_0)) - \cos^2(\angle(\psi_1, v_0))\Big| = \left|\cos^2\left(\frac{\pi/2 - \theta}{2}\right) - \cos^2\left(\frac{\pi/2 + \theta}{2}\right)\right| \\
&= \left|\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2} - \theta\right) - \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2} + \theta\right)\right]\right| = |\sin(\theta)| = \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2}
\end{aligned}$$

3. Define $V|\psi\rangle = \sum_i \sqrt{M_i}|\psi\rangle|i\rangle$. For any $|\phi\rangle, |\psi\rangle$,

$$\langle\phi| V^\dagger V |\psi\rangle = \sum_i \langle\phi| M_i |\psi\rangle = \langle\phi|\psi\rangle.$$

Thus V is an isometry on the subspace with zero ancilla and *extends* to a unitary U on the full space. Measuring the ancilla gives outcome i with probability $\left\|\sqrt{M_i}|\psi\rangle\right\|^2 = \langle\psi| M_i |\psi\rangle$. This extends to mixed state inputs by linearity.

4. Given U, set $K_i = (\mathbb{I} \otimes \langle i|)U(\mathbb{I} \otimes |0\rangle)$ and $M_i = K_i^\dagger K_i$. It remains to verify that $\{M_i\}$ is a valid POVM. Each M_i is positive semidefinite since, for every $|\phi\rangle$,

$$\langle\phi| M_i |\phi\rangle = \|K_i|\phi\rangle\|^2 \geq 0.$$

Moreover, using $\sum_i |i\rangle\langle i| = \mathbb{I}$ and $U^\dagger U = \mathbb{I}$, we obtain

$$\begin{aligned}\sum_i M_i &= \sum_i (\mathbb{I} \otimes \langle 0|)U^\dagger(\mathbb{I} \otimes |i\rangle\langle i|)U(\mathbb{I} \otimes |0\rangle) \\ &= (\mathbb{I} \otimes \langle 0|)U^\dagger U(\mathbb{I} \otimes |0\rangle) \\ &= \mathbb{I}.\end{aligned}$$

Furthermore,

$$\mathbf{Pr}[i] = \mathrm{tr}\left(K_i \rho K_i^\dagger\right) = \mathrm{tr}(M_i \rho).$$

5. The previous steps argue that a POVM $\{M, \mathbb{I} - M\}$ is the optimal measurement. Then, for any measurement M, the success probability can be expressed as $|\mathrm{tr}(M\rho_1) - \mathrm{tr}(M\rho_2)| = |\mathrm{tr}(M\rho)|$. Observe that $\mathrm{tr}(M\rho) = -\mathrm{tr}((\mathbb{I} - M)\rho)$ since $\mathrm{tr}(\rho) = 0$. Therefore, we can drop the absolute value in the optimization, yielding the displayed form.

6. Diagonalize $\rho = \sum_j \lambda_j |v_j\rangle\langle v_j|$. So $\Delta_{\mathrm{opt}} = \max_{0 \leq M \leq \mathbb{I}} \mathrm{tr}(M\rho)$.

$$\mathrm{tr}(M\rho) = \sum_j \lambda_j \langle v_j|M|v_j\rangle \leq \sum_{\lambda_j > 0} \lambda_j,$$

with equality for the projector onto the positive-eigenvalue subspace. As $\sum_j \lambda_j = 0$,

$$\boxed{\Delta_{\mathrm{opt}} = \sum_{\lambda_j > 0} \lambda_j = \frac{1}{2}\sum_j |\lambda_j| = \frac{1}{2}\mathrm{tr}\,|\rho|.}$$

7. For pure states, ρ has rank at most 2 and trace 0, so its nonzero eigenvalues are $\pm\lambda$ (including the

case $\lambda = 0$). Directly,

$$\operatorname{tr}\left(\rho^2\right) = \operatorname{tr}\left(\rho_0^2\right) + \operatorname{tr}\left(\rho_1^2\right) - 2\operatorname{tr}(\rho_0\rho_1) = 2\left(1 - |\langle\psi_0|\psi_1\rangle|^2\right) = 2\lambda^2.$$

Hence $\boxed{\operatorname{tr}\sqrt{\rho^2} = 2\sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2}}$.

8. One can verify that $\rho = |\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|$ is $\propto |v_0\rangle\langle v_0| - |v_1\rangle\langle v_1|$. Therefore, the positive eigenvalue space of ρ is the span of $|v_0\rangle$ and the negative eigenvalue space of ρ is the span of $|v_1\rangle$. The remaining $2^n - 2$ dimensional space has eigenvalue 0. This corresponds to the optimal POVM $\{|v_0\rangle\langle v_0|, |v_1\rangle\langle v_1|, \mathbb{I} - |v_0\rangle\langle v_0| - |v_1\rangle\langle v_1|\}$ which is a tripartite measurement corresponding to guessing 0, 1, and ?. It also matches the measurement from earlier.

Problem 6.

1. Write $|\psi\rangle = \sum_{y\in\{0,1\}} \alpha_y |y\rangle$. Recall that our Bell basis is defined by

$$|\Phi_{xz}\rangle = (X^x Z^z \otimes \mathbb{I})\,|\mathrm{EPR}\rangle, \qquad x, z \in \{0, 1\}.$$

We can verify that the initial state in the Bell basis of registers Q, A is

$$|\psi\rangle_Q |\mathrm{EPR}\rangle_{AB} = \frac{1}{2} \sum_{x,z\in\{0,1\}} |\Phi_{xz}\rangle_{QA} \otimes Z^z X^x |\psi\rangle_B .$$

Therefore, when Alice measures the Bell basis and obtains (x, z), Bob's post-measurement state is $Z^z X^x |\psi\rangle$. Bob applies the correction $X^x Z^z$ to recover the original state, since $X^x Z^z Z^z X^x = \mathbb{I}$. Thus teleportation succeeds for every measurement outcome.

2. Tracing out Eve's register gives

$$\rho_{AB} = \operatorname{tr}_E(|\mathrm{cat}\rangle\langle\mathrm{cat}|) = \frac{1}{2} \sum_{t\in\{0,1\}} |t, t\rangle\langle t, t| .$$

Thus Alice and Bob share a probabilistic mixture of $|00\rangle$ and $|11\rangle$ rather than an EPR pair.

Let $\rho_Q = |\psi\rangle\langle\psi|$. The initial density matrix of their three qubits is

$$\rho_{QAB} = \rho_Q \otimes \rho_{AB} = \frac{1}{2} \sum_{t\in\{0,1\}} \rho_Q \otimes |t\rangle\langle t|_A \otimes |t\rangle\langle t|_B .$$

Recall that Alice measures Q, A in the Bell basis $|\Phi_{xz}\rangle$. For a fixed value of t, the probability of

outcome (x, z) is

$$\begin{aligned} p_{xz|t} &= \operatorname{tr}\left(|\Phi_{xz}\rangle\langle\Phi_{xz}|\,(\rho_Q \otimes |t\rangle\langle t|)\right) \\ &= \frac{1}{2}\,\langle t \oplus x|\,\rho_Q\,|t \oplus x\rangle \\ &= \frac{1}{2}|\alpha_{t\oplus x}|^2. \end{aligned}$$

Here we used that $(\mathbb{I} \otimes \langle t|)\,|\Phi_{xz}\rangle$ is proportional to $|t \oplus x\rangle$. Bob's qubit is initially $|t\rangle$. After receiving (x, z), he applies $X^x Z^z$, which maps it to $|t \oplus x\rangle$ up to a global phase. Therefore, averaging over the two values of t and all measurement outcomes, Bob's final density matrix is

$$\begin{aligned} \rho_B &= \frac{1}{2}\sum_{t,x,z\in\{0,1\}} p_{xz|t}\,|t \oplus x\rangle\langle t \oplus x| \\ &= \frac{1}{2}\sum_{t,x\in\{0,1\}} |\alpha_{t\oplus x}|^2\,|t \oplus x\rangle\langle t \oplus x| \\ &= \boxed{\sum_{y\in\{0,1\}} |\alpha_y|^2\,|y\rangle\langle y|}. \end{aligned}$$

3. The density matrix Alice intended to send is

$$|\psi\rangle\langle\psi| = \sum_{x,y\in\{0,1\}} \alpha_x \alpha_y^*\,|x\rangle\langle y|\,.$$

Instead, Bob receives the measured version of the state Alice intended to send. In fact, Alice could produce exactly the same density matrix without using their shared state. She measures $|\psi\rangle$ in the computational basis and sends the resulting classical bit x to Bob, who prepares $|x\rangle$. Bob then receives $|x\rangle$ with probability $|\alpha_x|^2$, giving exactly the density matrix ρ_B.

4. Let

$$|\Psi\rangle = \sum_{x\in\{0,1\}} \alpha_x\,|x, x\rangle\,|g_x\rangle\,.$$

Its density matrix is

$$|\Psi\rangle\langle\Psi| = \sum_{x,y\in\{0,1\}} \alpha_x \alpha_y^*\,|x, x\rangle\langle y, y| \otimes\ |g_x\rangle\langle g_y|\,,$$

Recall that $\operatorname{tr}(|u\rangle\langle v|) = \langle v|u\rangle$. Therefore, tracing out the garbage register gives

$$\boxed{\rho_{12} = \sum_{x,y\in\{0,1\}} \alpha_x \alpha_y^*\,\langle g_y|g_x\rangle\,|x, x\rangle\langle y, y|\,.}$$

Equivalently, in the computational basis,

$$\rho_{12} = \begin{pmatrix} |\alpha_0|^2 & 0 & 0 & \alpha_0 \alpha_1^* \langle g_1 | g_0 \rangle \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_1 \alpha_0^* \langle g_0 | g_1 \rangle & 0 & 0 & |\alpha_1|^2 \end{pmatrix}.$$

If $|g_0\rangle = |g_1\rangle$, then $\langle g_0 | g_1 \rangle = 1$, and

$$\rho_{12} = |\phi\rangle\langle\phi|, \qquad |\phi\rangle = \sum_{x \in \{0,1\}} \alpha_x |x, x\rangle.$$

Thus the first two qubits remain in the intended pure state, since the garbage register is unentangled with them.

On the other hand, if $|g_0\rangle$ and $|g_1\rangle$ are orthogonal, then $\langle g_0 | g_1 \rangle = 0$, giving

$$\rho_{12} = \sum_{x \in \{0,1\}} |\alpha_x|^2 |x, x\rangle\langle x, x|.$$

The off-diagonal entries have vanished entirely. This is exactly what happened in Eve's attack: her qubit contains information about which computational-basis branch Alice and Bob share, and tracing it out produces a probabilistic mixture rather than the desired entangled pure state.