
Problem set 1 — Solutions

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Problem 1 (Tensor Products and Bra-ket notation). Let $\{|i\rangle\}$ denote the computational basis. By the definition of matrix entries, $A|j\rangle = \sum_i A_{ij}|i\rangle$, so $\langle i|A|j\rangle = A_{ij}$. Summing the diagonal entries yields $\text{tr}(A) = \sum_i \langle i|A|i\rangle$. Inserting the identity $\mathbb{I} = \sum_k |k\rangle\langle k|$ between A and C gives

$$\langle i|AC|j\rangle = \sum_k \langle i|A|k\rangle\langle k|C|j\rangle. \quad (1)$$

The definition $(A \otimes B)(|u\rangle \otimes |v\rangle) = A|u\rangle \otimes B|v\rangle$ shows that

$$(A \otimes B)(C \otimes D)(|u\rangle \otimes |v\rangle) = (AC|u\rangle) \otimes (BD|v\rangle), \quad (2)$$

which proves $(A \otimes B)(C \otimes D) = AC \otimes BD$, since product vectors span the space. Similarly,

$$(\langle i| \otimes \langle k|)(A \otimes B)(|j\rangle \otimes |\ell\rangle) = \langle i|A|j\rangle \langle k|B|\ell\rangle. \quad (3)$$

Taking conjugates of these entries, with row and column indices exchanged, proves $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$. Finally,

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = \mathbb{I}, \quad (4)$$

and the same holds in the opposite order, so $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Problem 2 (Tensor-product bases). (Here the first basis has d_1 elements, with indices $1, \dots, d_1$.) For any i, i', j, j' ,

$$(\langle a_i| \otimes \langle b_j|)(|a_{i'}\rangle \otimes |b_{j'}\rangle) = \langle a_i|a_{i'}\rangle \langle b_j|b_{j'}\rangle = \delta_{ii'}\delta_{jj'}. \quad (5)$$

Thus the $d_1 d_2$ product vectors are orthonormal. Since $\mathbb{C}^{d_1 d_2}$ has dimension $d_1 d_2$, they form a basis.

Problem 3 (Higher dimensional shenanigans). In n dimensions, every outer center is at distance $\sqrt{n}$ from the central center. Consequently, tangency (when the central ball fits without overlapping the outer balls) requires

$$r_n + 1 = \sqrt{n}, \quad \text{so} \quad \boxed{r_n = \sqrt{n} - 1}. \quad (6)$$

In particular, $\boxed{r_2 = \sqrt{2} - 1}$ and $\boxed{r_3 = \sqrt{3} - 1}$. The distance from the cube's center to each face is 2. Hence the central ball protrudes beyond the cube precisely if $r_n > 2$, i.e., $n > 9$. The smallest such dimension is therefore $\boxed{n_0 = 10}$ (at $n = 9$ it just touches the faces).

Problem 4 (Validity of Born's rule). Write $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$. Each Born probability $p(x) = |\alpha_x|^2$ is

nonnegative. Since $|\psi\rangle$ is normalized and the computational basis is orthonormal,

$$\sum_x p(x) = \sum_x |\alpha_x|^2 = \langle \psi | \psi \rangle = 1. \quad (7)$$

Thus Born's rule defines a probability distribution.

Problem 5 (Swap unitary). Acting on a computational basis vector $|a, b\rangle$, the three CNOTs (from right to left) produce

$$(a, b) \mapsto (a, a \oplus b) \mapsto (b, a \oplus b) \mapsto (b, a). \quad (8)$$

The second step holds because $a \oplus (a \oplus b) = b$. Therefore the circuit sends $|a, b\rangle$ to $|b, a\rangle$ for every $a, b \in \{0, 1\}$. By linearity, for any single-qubit states $|\psi\rangle = \sum_{a \in \{0,1\}} \alpha_a |a\rangle$ and $|\phi\rangle = \sum_{b \in \{0,1\}} \beta_b |b\rangle$,

$$\begin{aligned} \text{CNOT}_{12} \text{CNOT}_{21} \text{CNOT}_{12} (|\psi\rangle \otimes |\phi\rangle) &= \sum_{a,b \in \{0,1\}} \alpha_a \beta_b |b, a\rangle \\ &= \left(\sum_{b \in \{0,1\}} \beta_b |b\rangle \right) \otimes \left(\sum_{a \in \{0,1\}} \alpha_a |a\rangle \right) \\ &= |\phi\rangle \otimes |\psi\rangle. \end{aligned} \quad (9)$$

Thus, the circuit agrees with the definition of SWAP on all tensor product states.

Problem 6 (Probability theory review). 1. For $k > 0$, take $Z = k$ with probability p and $Z = 0$ otherwise, for any $0 \leq p \leq 1$. Then $\Pr[Z \geq k] = p = \mathbb{E}[Z]/k$. At $k = 0$, the displayed Markov bound is undefined; for $k < 0$ the usual Markov inequality is not valid. Thus the requested statement needs the assumption $k > 0$.

2. For $t > 0$, take $Z = t$ and $Z = -t$ with probability $1/2$ each. Then $\mathbb{E}[Z] = 0$, $\text{Var}(Z) = t^2$, and $\Pr[|Z - \mathbb{E}[Z]| \geq t] = 1 = \text{Var}(Z)/t^2$.

3. For $0 \leq \delta \leq 1$, take logarithms to reduce the desired bound to

$$(1 + \delta) \ln(1 + \delta) - \delta \geq \frac{\delta^2}{3}. \quad (10)$$

Indeed, letting $h(\delta)$ be the left side minus the right side, we have $h(0) = 0$ and

$$h'(\delta) = \ln(1 + \delta) - \frac{2\delta}{3} \geq \delta - \frac{\delta^2}{2} - \frac{2\delta}{3} = \frac{\delta(2 - 3\delta)}{6}, \quad (11)$$

which by itself only handles $\delta \leq 2/3$. For the whole interval, instead use $\ln(1 + \delta) \geq 2\delta/(2 + \delta)$

(differentiate both sides). Then

$$h'(\delta) \geq \frac{2\delta}{2+\delta} - \frac{2\delta}{3} = \frac{2\delta(1-\delta)}{3(2+\delta)} \geq 0. \quad (12)$$

Exponentiating proves the claim. The restriction $\delta \leq 1$ is needed: the stated inequality is false for sufficiently large δ (the usual general upper-tail exponent is $\mu[(1+\delta)\ln(1+\delta) - \delta]$).

Graded problems

Problem 7 (Partial measurements). Write $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$.

1. To measure qubit j , swap qubits 1 and j , measure the first qubit, and swap them back. This has the same outcome probabilities and post-measurement state as directly measuring qubit j . Indeed, if S_{1j} denotes the unitary swapping qubits 1 and j , then the projector onto qubit j having value b is

$$\underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{j-1 \text{ qubits}} \otimes |b\rangle\langle b| \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{n-j \text{ qubits}} = S_{1j}^\dagger (|b\rangle\langle b| \otimes \mathbb{I}_{2^{n-1}}) S_{1j}. \quad (13)$$

2. For $y = y_1 \cdots y_k$, define the unnormalized residual vector

$$|\phi_y\rangle \stackrel{\text{def}}{=} (\langle y_1 \cdots y_k | \otimes \mathbb{I}_{2^{n-k}}) |\psi\rangle. \quad (14)$$

We claim that the state after measuring the first k qubits and observing outcomes $y_1, \dots, y_k$ is

$$\boxed{|y_1 \cdots y_k\rangle \otimes \frac{|\phi_y\rangle}{\| |\phi_y\rangle \|}}, \quad (15)$$

provided that $|\phi_y\rangle$ is nonzero.

To see this, recall from the measurement axiom for the first qubit that after measuring the first qubit and obtaining y_1 , the remaining $n-1$ qubits are in the unit state proportional to $(\langle y_1 | \otimes \mathbb{I}_{2^{n-1}}) |\psi\rangle$. The first qubit is now unentangled from the rest of the system. Measuring the second qubit and applying the same observation, the remaining $n-2$ qubits are in the unit state proportional to $(\langle y_1, y_2 | \otimes \mathbb{I}_{2^{n-2}}) |\psi\rangle$. Continuing inductively, the state of the remaining $n-k$ qubits is the unit vector parallel to $|\phi_y\rangle$, while the first k qubits are in state $|y_1 \cdots y_k\rangle$. This proves the claimed expression for the final state.

Notice that we have performed a small mathematical trick: we kept track of the unnormalized residual vector and only normalized at the very end. If we had normalized after each measurement, the normalization factor at step i would have been $\Pr[y_i \mid y_1 \cdots y_{i-1}]^{-1/2}$. Consequently, the total

normalization factor is

$$\frac{1}{\|\phi_y\rangle\|} = \prod_{i=1}^k \Pr[y_i \mid y_1 \cdots y_{i-1}]^{-1/2}, \quad (16)$$

where for $i = 1$ we interpret the conditional probability as $\Pr[y_1]$. Squaring and applying the chain rule for conditional probabilities, we conclude that

$$\Pr[y_1 \cdots y_k] = \prod_{i=1}^k \Pr[y_i \mid y_1 \cdots y_{i-1}] = \|\langle y_1 \cdots y_k \mid \otimes \mathbb{I}_{2^{n-k}} \mid \psi \rangle\|^2. \quad (17)$$

3. Take $k = n$. The residual register is one-dimensional and the measurement returns $x \in \{0, 1\}^n$ with probability $|\alpha_x|^2$, leaving the state $|x\rangle$ (for nonzero-probability outcomes). This is the full computational-basis measurement.
4. For an outcome b on qubit j , let $P_{j,b} = \mathbb{I} \otimes \cdots \otimes |b\rangle\langle b| \otimes \cdots \otimes \mathbb{I}$, a projection matrix whose action is only non-trivial on the j -th qubit/tensor product. Projectors acting on different qubits/tensor products commute. Thus the product $P = \prod_{j \in S} P_{j,y_j}$ is independent of the order, as are its squared norm $\|P|\psi\rangle\|^2$ (the joint probability) and its normalized action $P|\psi\rangle / \|P|\psi\rangle\|$ (the conditioned state).

Problem 8 (A very large Hilbert space). Let $N = 2^n = \dim \mathcal{H}$.

1. The maximal number is $\boxed{N = 2^n}$: orthogonal nonzero vectors are linearly independent, and an orthonormal basis attains the bound.
2. We note that $\langle \psi_f | \psi_g \rangle$ is proportional to a sum of 2^n coin flips. To see this define $X_x \in \{0, 1\}$ as the random variable $1[f(x) = g(x)]$. Then,

$$\langle \psi_f | \psi_g \rangle = \frac{1}{2^n} \sum_x (-1)^{f(x)+g(x)} \quad (18)$$

$$= \frac{1}{2^n} \sum_x 2X_x - 1 \quad (19)$$

$$= -1 + \frac{1}{2^{n-1}} \sum_x X_x. \quad (20)$$

Therefore, by a Chernoff bound,

$$\Pr \left[|\langle \psi_f | \psi_g \rangle| > \epsilon \right] = \Pr \left[\left| \sum_x X_x - 2^{n-1} \right| > \epsilon 2^{n-1} \right] \leq 2 \exp \left(-\frac{\epsilon^2 2^n}{6} \right). \quad (21)$$

3. Our greedy strategy will be to start with a set S consisting of all phase states. While the set S is not vacuous, select a phase state $|\psi_f\rangle$ and remove all states from S that are not ϵ -orthogonal to $|\psi_f\rangle$. By

construction, the selected phase states will be ϵ -orthogonal to all prior selected phase states. This generates a pairwise- ϵ -orthogonal set. Each pruning of S removes at most $p \cdot 2^{2^n}$ phase states where p is the bound from the previous subproblem. Therefore, our selected set has size at least $1/p$ where p is the bound in eq. (21).

Problem 9 (Reversible classical computation).

1. The matrix T is $\mathbb{I}_8$ except the bottom right corner $\mathbb{I}_2$ is replaced with X . Equivalently,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (22)$$

2. Take the circuit C for f and from the partial ordering of gates, assign a consistent total ordering to get gates $g_1, \dots, g_{|f|}$. The AND gates from this circuit can be replaced with a reversible AND gate and a fresh ancilla, the COPY gates with a CNOT gate and a fresh ancilla, and the NOT gates stay the same. Using CNOT gates, the value of the output wires of $f(x)$ can be added to the wires containing y . All fresh ancilla wires initialized classify as the junk.

Next, we simply run the previous reversible circuit in reverse except for the final CNOT gates used to copy the answer. Since all gates are self-inverse, this will systematically uncompute the values in the ancilla wires till they reset to zeroes.

3. Each gate of f appears twice in the compute and the uncompute. Additionally, m CNOT gates were used to copy the answer onto y . So, $\boxed{2|f| + m}$.