Problem set 3

CSE 534 Autumn 2026

Due date: October 26th 11:59 pm

Instructions: Solutions should be legibly handwritten or typeset. Mathematically rigorous solutions are expected for all problems unless explicitly stated. You are encouraged to collaborate on problems in small teams; however, each team member must write and submit their own individual solution. Read the AI tools policy on the course webpage and adhere to it. For grading purposes, start each problem on a new page.

Practice problems (not graded)

These problems are not graded. Do not submit a solution, but do ensure that you can solve the problem.

Problem 1 (Indistinguishable states)**.** When can we distinguish quantum states?

1. Let $|\psi\rangle$ and $|\psi'\rangle$ be orthogonal single qubit states. Show that

$$\frac{1}{\sqrt{2}}|00\rangle + |11\rangle = \frac{|\psi, \psi^\star\rangle + |\psi', \psi'^\star\rangle}{\sqrt{2}}.$$

2. Let $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$. Show that the states $|\phi\rangle$ and $e^{i\theta}|\phi\rangle$ for $\theta \in \mathbb{R}$ cannot be distinguished by any combination of our 'axioms of quantum computation'.
Hint: Consider the corresponding density matrices.

3. Consider the following two random preparations of a qubit.

   (a) With probability 1/3, prepare $|0\rangle$, and with probability 2/3, prepare $|+\rangle$.

   (b) With probability 1/3, prepare $|1\rangle$, and with probability 2/3, prepare $|-\rangle$.

   Compute the density matrix of each ensemble. Are they equal? If not, identify any basis measurement $\{|b_0\rangle, |b_1\rangle\}$ that differentiates them (it doesn't have to be optimal).

Problem 2 (Expectation of an operator)**.** In practice, we care about the outcome of a quantum system averaged over many trials. Consider a qubit $|\psi\rangle \in \mathbb{C}^2$ and associate the measurement $|0\rangle$ with $+1$ and a measurement of $|1\rangle$ with -1.

1. Show the expectation of this experiment is $\langle\psi|Z|\psi\rangle$ where

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

2. This gives rise to the notation, $\langle Z \rangle_\psi = \langle \psi | Z | \psi \rangle$ (or $\langle Z \rangle$ when the state ψ is clear from context). Give an experiment with expectation $\langle X \rangle$ where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. What is the appropriate definition of $\langle Z \rangle_\rho$ for a density matrix ρ? (No explanation required).

Graded problems

Problem 3 (Density matrices of the W state)**.** The W_n state is an entangled state of n qubits defined as:

$$\begin{aligned} |W_n\rangle &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underbrace{|0\rangle \ldots |0\rangle}_{j-1 \text{ times}} |1\rangle \underbrace{|0\rangle \ldots |0\rangle}_{n-j \text{ times}} \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j |0^n\rangle . \end{aligned}$$

Here X_j is the X bit-flip operator applied to the j-th qubit.

1. **(2 points)** What is the reduced density matrix of the W state on 1 qubit?

2. **(1 points)** Consider the W state for $n = 3$. What is the reduced density matrix on any two of the qubits out of three?

Problem 4. Consider a device that ideally produces the state ρ_0 but due to manufacturing defects, produces the state ρ_1. We will show that if ρ_0 and ρ_1 are close, then no quantum process can distinguish these two devices with high probability. For any process P, quantify how well it distinguishes ρ_0 and ρ_1 by:

$$\Delta \stackrel{\text{def}}{=} |\mathbf{Pr}(P(\rho_0) \text{ outputs } 0) - \mathbf{Pr}(P(\rho_1) \text{ outputs } 0)|.$$

Solve the first two parts of this problem individually. For Parts 1 and 2, assume that $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho_1 = |\psi_1\rangle\langle\psi_1|$ are pure states.

1. **(1 point)** Consider the simplest strategy: measure in a basis for which $|\psi_0\rangle$ is a basis vector and guess 0 if the measurement is $|\psi_0\rangle$ and 1 otherwise. Show that then

$$\Delta = 1 - |\langle \psi_0 | \psi_1 \rangle|^2.$$

2. **(3 points)** This strategy is not optimal. Find a better measurement for which

$$\Delta = \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2}.$$

(Hint: There is a 2-dimensional space containing $|\psi_0\rangle$ and $|\psi_1\rangle$. It may be useful to remember the trigonometric identities of $2\sin x \sin y = \cos(x-y) - \cos(x+y)$ and $\cos 2x = 2\cos^2 x - 1$.)

We will show that this second strategy is indeed optimal for pure states, and more generally characterize the optimal distinguishing probability for arbitrary density matrices. To show the upper bound, we will first introduce a generalized form of measurement called a *positive-operator valued measurement* (POVM). A POVM is a set of Hermitian positive semidefinite operators $\{M_i\}$ on a Hilbert space $\mathcal{H}$ that sum up to identity

$$\sum_{i=1}^{n} M_i = \mathbb{I}_{\mathcal{H}}.$$

The probability of measuring outcome i on a state ρ is given by $\mathbf{Pr}(i) = \mathrm{tr}(M_i\rho)$. This generalizes a basis measurement as we can consider $M_i = |b_i\rangle\langle b_i|$ for any basis $\{|b_i\rangle\}$. An important difference between basis measurements and POVMs is that the elements of a POVM are not necessarily orthogonal nor projectors; therefore, the number of elements can be larger than the dimension of the Hilbert space $\mathcal{H}$.

Instead, POVMs are exactly as descriptive as applying a unitary U to the state and ancilla $\rho\otimes|0\ldots0\rangle\langle0\ldots0|$ followed by a measurement of some of the qubits.

3. **(1 point)** For any POVM $\{M_i\}$, let $A_i = \sqrt{M_i}$, consider the following partial transformation:

$$U \,:\, |\psi\rangle\,|0\rangle_{\mathsf{ancilla}} \mapsto \sum_{i=1}^{n} A_i\,|\psi\rangle\,|i\rangle_{\mathsf{ancilla}}\,.$$

   Conclude that U is unitary and that if U is followed by a measurement of the ancilla register, this produces the same statistics as the POVM.

4. **(2 points)** Given a unitary U acting on the state and some ancilla of dimension n initialized to zero, construct a POVM equivalent to applying U and measuring the ancilla in the standard basis.

Returning to the problem at hand, we can generalize the distinguishing measurement as a POVM with two elements M and $\mathbb{I} - M$, with the two outcomes corresponding to answering 0 and 1, respectively.

For partial credit on the remaining parts, you may assume that ρ_0 and ρ_1 are pure states.

5. **(2 points)** Show that then the optimal value of Δ is

$$\Delta_{\mathrm{opt}} = \max_{0\le M\le\mathbb{I}} \mathrm{tr}\,(M\rho)$$

   where $\rho = \rho_0 - \rho_1$.

6. **(3 points)** Conclude that

$$\max_{0\le M\le\mathbb{I}} \mathrm{tr}\,(M\rho) = \frac{1}{2}\,\mathrm{tr}\,|\rho|.$$

Recall, $|\rho|$ is the Hermitian matrix with the same eigenbasis as ρ but each eigenvalue has been replaced with its absolute value. (Hint: Consider an optimal M in the basis where ρ is diagonal).

7. **(2 points)** In the case where $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho_1 = |\psi_1\rangle\langle\psi_1|$ are pure, finish by showing

$$\operatorname{tr}|\rho| = 2\sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2}.$$

(Hint: ρ is a rank 2 matrix; therefore it has only 2 non-zero eigenvalues.)

8. **(Optional)** In the pure-state case, give a justification as to why the maximizing M and the measurement you gave in Part 2 are the same.

Problem 5 (Quantum teleportation and unwanted entanglement)**.** Alice wants to send an unknown single-qubit pure state $|\psi\rangle$ to Bob. However, Alice and Bob are far apart, and Alice can only communicate with Bob by sending classical bits. Surprisingly, if Alice and Bob share an entangled pair of qubits, they can accomplish this task by communicating only two classical bits. This is known as *quantum teleportation*, a dual problem to superdense coding that we observed in lecture. Suppose Alice and Bob initially share the EPR pair

$$|\mathrm{EPR}\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

where Alice holds qubit A and Bob holds qubit B. Alice additionally holds a qubit Q in the unknown state $|\psi\rangle$. The combined system is initially in the state

$$|\psi\rangle_Q \otimes |\mathrm{EPR}\rangle_{AB}.$$

Alice and Bob execute the following protocol:

- Alice measures qubit Q and qubit A in the Bell basis from lecture to obtain bits (x, z).
- Bob applies a correction based on x and z to recover the state $|\psi\rangle$ on qubit B.

1. **Quantum teleportation. (2 points)** Describe Bob's correction and prove its correctness.

2. **An untrustworthy source of entanglement. (2 points)** Now suppose that Alice and Bob obtain their entangled qubits from Eve. Eve claims to have prepared an EPR pair, but instead prepares the three-qubit cat state

$$|\mathrm{cat}\rangle_{ABE} = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

Eve gives qubit A to Alice and qubit B to Bob, while secretly keeping qubit E.

Suppose Alice and Bob do not realize Eve's deception and execute the teleportation protocol from Part 1. Calculate the density matrix ρ_B of Bob's qubit after he applies his correction.

3. **Has teleportation succeeded? (2 points)** Compare ρ_B to the density matrix of the state Alice intended to transmit $|\psi\rangle\langle\psi|$.

   Describe a different procedure that would have produced the same density matrix ρ_B for Bob but would not have "used up" their shared entangled state.

4. **Connection to uncomputing garbage. (2 points)** Consider a quantum computation that starts in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and during copying produces a garbage register G, leaving the joint state

   $$\alpha|0,0\rangle|g_0\rangle + \beta|1,1\rangle|g_1\rangle,$$

   where $|g_0\rangle$ and $|g_1\rangle$ are normalized states of the garbage register.

   Calculate the reduced density matrix of the first two qubits after tracing out G. Show that its off-diagonal entries depend on $\langle g_0|g_1\rangle$.

   Explain what happens when the garbage states are identical and when they are orthogonal. Relate your answer to Eve's attack on quantum teleportation.

5. **Why uncomputing garbage matters.** *In Problem Set 1, we saw how to implement classical computations reversibly and then "uncompute" the intermediate garbage. We can now understand why this is important when a quantum algorithm uses a classical computation as a subroutine. A classical circuit might compute the correct answer $f(x)$ on every input x, but when run on a superposition of inputs, it can leave behind garbage that depends on x. This garbage becomes entangled with the registers we care about. Even if we never measure the garbage, the resulting quantum state may behave differently from the state we intended. As we saw in this problem, when different computational branches leave distinguishable garbage states, the off-diagonal entries of the reduced density matrix become smaller or even vanish entirely. This changes the probabilities of subsequent measurements and can destroy the quantum interference needed by the algorithm. Eve's cat state illustrates the same phenomenon: Alice and Bob share the correct classical correlations, but these are not sufficient for quantum teleportation. By uncomputing the garbage, we return the auxiliary registers to a fixed state, independent of the computational input. The garbage is then no longer entangled with the registers we care about, allowing the algorithm to continue using quantum interference as intended.*