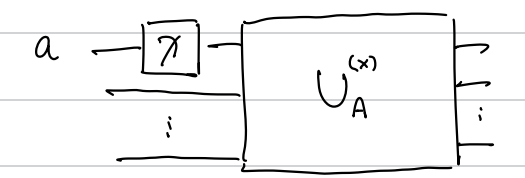
Since, we are considering <u>all</u> CHSH strategies regardless of size, $\exists$ an equivalent strategy for Alice with exactly 1 meas. and its at the end for the outcome.

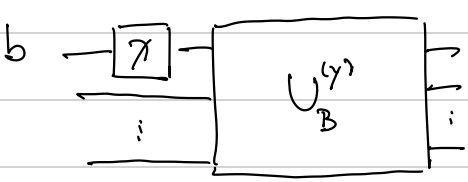
Alice:





P_{AB}.

Bob:



Define $A_x^i = U_A^{x\dagger} \left(|i\rangle\langle i| \otimes \mathbb{1} \right) U_A^x$, a projector.

$A_x^i \geq 0$ and $A_x^0 + A_x^1 = \mathbb{1}$.

$\{A_x^0, A_x^1\}$ form a POVM.

$\Pr[\text{Alice outputs } i \text{ on input } x] = \mathrm{tr}\left(A_x^i \rho_A \right)$.

Generalized measurements (Interlude)

not necessarily projectors

POVM $\{Q_a\}$ s.t. $Q_a \geq 0$, $\sum_a Q_a = \mathbb{1}$.

$\Pr(a) = \mathrm{tr}(Q_a \rho)$. Note POVMs don't specify the post-measurement state.

Since $Q_a \geq 0 \Rightarrow \langle\psi|Q_a|\psi\rangle \geq 0 \ \forall\ |\psi\rangle$

$$\Rightarrow \mathrm{tr}(Q_a \rho) \geq 0.$$

$$\sum_a \Pr(a) = \sum_a \mathrm{tr}(Q_a \rho) = \mathrm{tr}\left(\left(\sum_a Q_a\right)\rho\right) = 1.$$

Possible implementations: $M_a^\dagger M_a = Q_a$

then $$\rho_a = \frac{M_a \rho M_a^\dagger}{\mathrm{tr}(Q_a \rho)}.$$

Ex. Fix $\eta \in [0,1]$. 2 outcome POVM

$$Q_\pm = \frac{\mathbb{1} \pm \eta Z}{2}. \quad \text{Eigenvalue of } Q_\pm \text{ are } \frac{1 \pm \eta}{2} \text{ so } Q_\pm \geq 0.$$

$$Q_+ + Q_- = \mathbb{1}.$$

$$Q_\pm = \frac{1 \pm \eta}{2} |0\rangle\langle 0| + \frac{1 \mp \eta}{2} |1\rangle\langle 1|.$$

$$\Pr[\text{outcome } +] = \operatorname{tr}(Q_+ \rho) = \frac{1}{2} + \frac{\eta}{2} \operatorname{tr}(Z\rho).$$

What is the operational interpretation of this POVM?

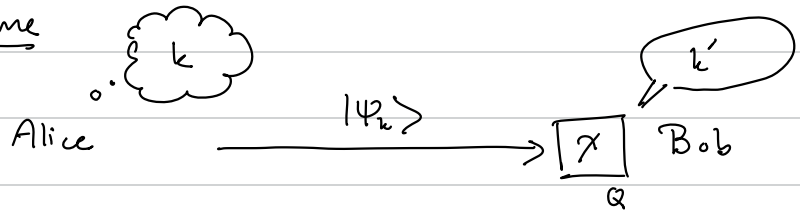
Ex. $\omega = e^{2\pi i/3}$ and define for $k = 0, 1, 2,$

$$|\psi_k\rangle := \frac{|0\rangle + \omega^k |1\rangle}{\sqrt{2}}. \qquad \text{Let } Q_k = \frac{2}{3} |\psi_k\rangle\langle\psi_k|.$$

$$= \frac{1}{3} \begin{pmatrix} 1 & \omega^k \\ \omega^{-k} & 1 \end{pmatrix}$$

Since $1 + \omega + \omega^2 = 0,$ $\sum Q_k = \mathbb{1}.$

Game



$$\Pr[k = k'] = \langle \psi_k | Q_k | \psi_k \rangle = \frac{2}{3}.$$

Turns out best possible algorithm (not easy to prove).

Implementing POVMs Pset problem!

Returning to CHSH. For general strategy,

$$p(a,b \mid x,y) = \mathrm{tr}\left(A_x^a \otimes B_y^b \quad \rho_{AB} \right).$$

So,

$$\Pr[\text{win}] = \frac{1}{4} \sum_{x,y} \sum_{a,b} \mathbb{1}[a \oplus b = xy] \, \mathrm{tr}\left(A_x^a \otimes B_y^b \quad \rho_{AB} \right).$$

Simplifying even further.

$$0 \mapsto +1, \quad 1 \mapsto -1, \qquad x \mapsto (-1)^x.$$

Assume Alice & Bob answer with $+1$ or -1. (Semantic)

Win condition is $a \cdot b = (-1)^{xy}$ and

Alice described by POVM $\{A_x^1, A_x^{-1}\}$. Sim. for Bob.

Define $A_x = A_x^1 - A_x^{-1} = 1 \cdot A_x^1 + (-1) \cdot A_x^{-1}$.

$A_x^2 = \mathbb{1}$ and so has only $+1, -1$ eigenvalues (binary observable).

$B_y = B_y^1 - B_y^{-1}$. Then,

$$\Pr[\text{win}] = \frac{1}{4} \sum_{x,y \in \{0,1\}} \sum_{a,b \in \{\pm 1\}} \left(\frac{1}{2} + \frac{ab(-1)^{xy}}{2} \right) \operatorname{tr}\left(A_x^a \otimes B_y^b \, \rho_{AB} \right)$$

$$= \frac{1}{2} + \frac{1}{8} \sum_{\substack{x,y \\ a,b}} ab(-1)^{xy} \operatorname{tr}\left(A_x^a \otimes B_y^b \, \rho_{AB} \right)$$

$$= \frac{1}{2} + \frac{1}{8} \sum_{\substack{x,y \\ a,b}} \operatorname{tr}\left((-1)^{xy} \; a A_x^a \otimes b B_y^b \; \rho_{AB} \right)$$

$$= \frac{1}{2} + \frac{1}{8} \left(\sum_{x,y} (-1)^{xy} \; A_x \otimes B_y \; \rho_{AB} \right).$$

Define $CHSH = \sum_{x,y} (-1)^{xy} \; A_x \otimes B_y$, an operator on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Then $\Pr[\text{win}] = \frac{1}{2} + \frac{1}{8} \operatorname{tr}(CHSH \; \rho_{AB})$.

For $T, S \in \mathcal{L}(\mathcal{H})$, the operator norm is defined as

$$\|T\|_{op} = \max_{\||\psi\rangle\| = 1} \|T|\psi\rangle\|.$$

Note $\|aT + S\|_{op} \leq |a| \cdot \|T\|_{op} + \|S\|_{op}$.

$$\|TS\|_{op} \leq \|T\|_{op} \cdot \|S\|_{op}$$

$$\|S \otimes T\|_{op} = \|S\|_{op} \cdot \|T\|_{op}$$

Claim ρ_{AB} dm. then

$$\mathrm{tr}(\mathrm{CHSH}\,\rho_{AB}) \leq \|\mathrm{CHSH}\|_{op}$$

$\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ then

$$\mathrm{tr}(\mathrm{CHSH}\,\rho_{AB}) = \sum_i p_i \langle\psi_i|\mathrm{CHSH}|\psi_i\rangle \leq \sum_i p_i \|\mathrm{CHSH}\|_{op} = \|\mathrm{CHSH}\|_{op}.$$

So suffices to prove that $\|\mathrm{CHSH}\|_{op} \leq 2\sqrt{2}$

Lemma For any binary observables A_0, A_1, B_0, B_1, $\|\mathrm{CHSH}\|_{op} \leq 2\sqrt{2}$.

Pf. CHSH

$$= A_0 \otimes B_0 + A_1 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_1$$
$$= (A_0 + A_1) \otimes B_0 + (A_0 - A_1) \otimes B_1$$

Using $A_0^2 = A_1^2 = B_0^2 = B_1^2 = \mathbb{1}$ + commutation then

$$CHSH^2 = (A_0 + A_1)^2 \otimes \mathbb{1} + (A_0 - A_1)^2 \otimes \mathbb{1}$$
$$+ (A_0 + A_1)(A_0 - A_1) \otimes B_0 B_1 + (A_0 - A_1)(A_0 + A_1) \otimes B_1 B_0$$

$$= 4\mathbb{1} + (A_0 A_1 + A_1 A_0 - A_0 A_1 - A_1 A_0) \otimes \mathbb{1}$$
$$+ (A_0 A_1 - A_1 A_0) \otimes (-B_0 B_1)$$
$$+ (A_0 A_1 - A_1 A_0) \otimes (B_1 B_0).$$

$$= 4\mathbb{1} + [A_0, A_1] \otimes [B_1, B_0]$$

where $[R, S] = RS - SR$ is short form for the commutator.

Since $\|A_i\|_{op} = 1$, then $\|[A_0, A_1]\|_{op} \leq 2$

and sim. $\|[B_1, B_0]\|_{op} \leq 2$.

Then $\|CHSH\|_{op}^2 = \|CHSH^2\|_{op} \leq \|4\mathbb{1}\|_{op} + \|[A_0, A_1]\|_{op} \cdot \|[B_1, B_0]\|_{op}$
$\leq 4 + 2 \cdot 2$
$= 8$

$\Rightarrow \|CHSH\|_{op} = 2\sqrt{2}$.

So indeed, no better strategy exists! Optimal value of the game is $\cos^2 \pi/8$.

Next: CHSH Rigidity.

Any strategy can be expressed as $(\{A_x\}, \{B_y\}, \rho_{AB})$.

Our goal is to characterize every strategy that succeeds w pr $\cos^2 \pi/8$.

Our optimal strategy:

$$A_0^* = Z, \quad A_1^* = X, \quad B_0^* = H, \quad B_1^* = \tilde{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$A_0^* A_1^* = -A_1^* A_0^*$, $B_0^* B_1^* = -B_1^* B_0^*$ anti-commutation

$CHSH^* = CHSH(A_0^*, A_1^*, B_0^*, B_1^*)$ has a unique $2\sqrt{2}$ eigenvector:

$|EPR\rangle$.