Maximally entangled and mixed states

$\mathcal{H}_A \cong \mathcal{H}_B = (\mathbb{C}^2)^{\otimes n}$ the dim. $\rho_A = \mathbb{1}/2^n$ is the maximally mixed state.

- In any basis $|b_1\rangle, \ldots, |b_d\rangle$, the prob of outcome j is 2^{-n} maximal entropy.

- Ideal for randomness generation, if we can certify Alice measures $\rho_A = \mathbb{1}/2^n$, we don't need to worry about what basis she measures in.

- Find a state $|\Phi\rangle_{AB}$ s.t. $\rho_A = \mathrm{tr}_B(|\Phi\rangle\langle\Phi|)$
  $|\Phi\rangle$ will be a maximally entangled state.

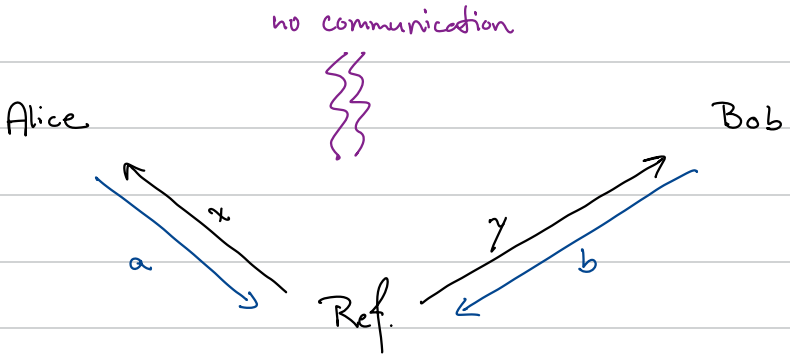
- Construct a test/game between Alice & Bob that can only* be won if the states shared is $|\Phi\rangle$.

We will solve this for $n=1$, $\rho_A = \mathbb{1}/2$ and $|\Phi\rangle = |EPR\rangle$.

The CHSH Game

- Power of entanglement
- Characterization of q. devices
- Certifiable randomness generation.

Setup:



Ref samples $x, y \in \{0, 1\}$ and sends over questions.

Alice & Bob respond with answers a, b, resp.

They "win" if $a \oplus b = x \cdot y$.

What is the max success prob. they can achieve?

Thm

Classical strategies $= 3/4$.

Q. strategies $= \cos^2 \pi/8 \sim 0.878$.

Classical: If $a = b = 0$ regardless of questions

then the win whenever $x \cdot y \neq 1$ (pr $3/4$).

Deterministic strategy.

$a, b : \{0,1\} \to \{0,1\}$.

x	y	condition
0	0	$0 = a(0) \oplus b(0)$
0	1	$0 = a(0) \oplus b(1)$
1	0	$0 = a(1) \oplus b(0)$
1	1	$1 = a(1) \oplus b(1)$

$1 = 0$

↑

Contradiction.

Shared randomness:

$$a, b : \{0,1\} \times \Omega \longrightarrow \{0,1\}.$$

$$\Pr[\text{win}] = \sum_{r \in \Omega} \Pr[r] \cdot \mathbb{1}\left[a(x,r) \oplus b(x,r) = xy\right]$$

$$\leq \max_{r} \mathbb{1}\left[a(x,r) \oplus b(x,r) = xy\right]$$

$$= 3/4.$$

Sharing entanglement?

Share an EPR pair.

Key ideas:

① Measurements in similar bases will agree w.h.p.

② Alice and Bob can choose a basis based on their question x or y.

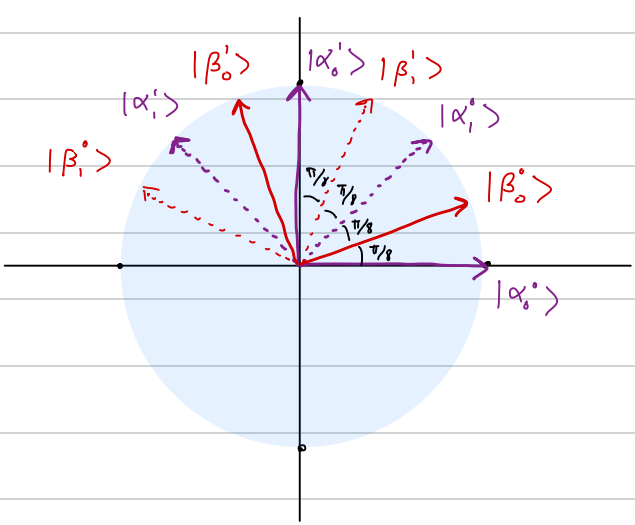
Bases at an angle: For any $\theta \in [0, \pi/2]$,

$$B_\theta := \{ \underbrace{|\theta\rangle := R_\theta |0\rangle}_{\text{outcome } 0}, \underbrace{|\theta^\perp\rangle := R_{\theta+\pi/2}|0\rangle}_{\text{outcome } 1} \}.$$

(Optimal) Q. strategy

Alice: on input $x = 0$, measures with B_0
on input $x = 1$, measures with $B_{\pi/4}$

Bob: on input $y = 0$, measures with $B_{\pi/8}$
on input $y = 1$, measures with $B_{-\pi/8}$.



$$\Pr[\text{win}] = \frac{1}{4} \sum_{x,y} \sum_{a,b} \mathbb{1}[a \oplus b = xy] \cdot p(a,b \mid x,y)$$

Lem If Alice measures in B_θ and Bob measures in B_ϕ, the $|EPR\rangle$ state for outcomes a and b, resp., then,

$$\Pr[a=b] = \cos^2(\theta-\phi), \quad \Pr[a \neq b] = \sin^2(\theta-\phi).$$

Pf. $\Pr[a=b=0] = |\langle \theta, \phi \mid EPR\rangle|^2$

$$\Pr[a=b=1] = |\langle \theta^\perp, \phi^\perp \mid EPR\rangle|^2$$

$$\langle \theta, \phi \mid EPR\rangle = \frac{1}{\sqrt{2}}\left(\langle \theta,\phi \mid 0,0\rangle + \langle \theta,\phi \mid 1,1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos\theta \cdot \cos\phi + \sin\theta \sin\phi\right)$$

$$= \frac{1}{\sqrt{2}} \cos(\theta-\phi).$$

Similarly, $\langle \theta^\perp, \phi^\perp \mid EPR\rangle = \frac{1}{\sqrt{2}} \cos(\theta-\phi)$. □

Q. strategy analysis.

x	y	angle between bases	win pr
0	0	$\pi/8$	$\cos^2 \pi/8$
0	1	$\pi/8$	$\cos^2 \pi/8$
1	0	$\pi/8$	$\cos^2 \pi/8$
1	1	$3\pi/8$	$\cos^2 \pi/8$

Implications

① Experimental way to disprove classical mechanics.

Get Alice & Bob to play n times.

If their win prob is $\gg 3/4$ as $n \to \infty$, suggests some other theory is at play. Q. is one plausible explanation.

② Disprove the "local" picture of nature.

$$\mathrm{tr}_A\left(|EPR\rangle\langle EPR|\right) = \mathrm{tr}_B\left(|EPR\rangle\langle EPR|\right) = \frac{\mathbb{1}}{2}.$$

A local description of the universe would require that $\mathbb{1}/2$ is a sufficient description of Alice's system. However, we know that $\mathbb{1}/2$ is equal to classical randomness.

Put it another way, the description of the state of a system cannot be described as the sum of the description of its parts.

The origin of the name "nonlocal game".

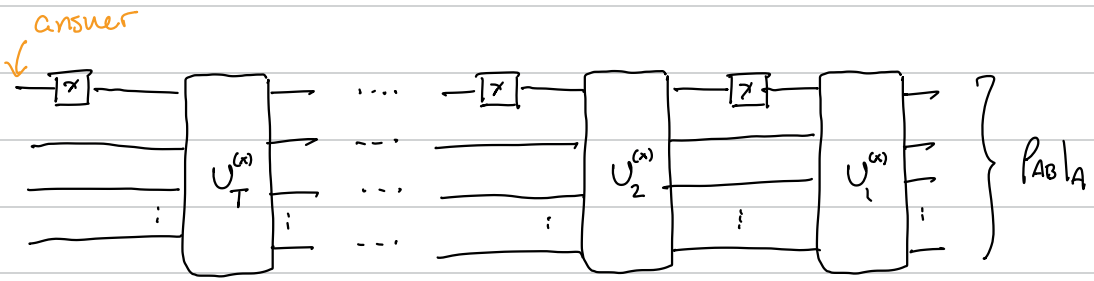
Can we do better?

Ans: No. The listed strategy is optimal. In fact, it is the only* strategy (assuming quantum) that can achieve $\cos^2 \pi/8$.

How do we prove this? First we need a way to characterize all q. strategies.

Starting state: $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Alice's general strategy:



Cumbersome! Find an equivalent strategy that is simpler to analyze. First, remove the intermediate measurements.

principle of deferred measurement:

Let $\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$ be the state of a system with the

the first qubit identified.

Measuring the qubit transforms the d.m. to

$$\sum_{k \in \{0,1\}} |k\rangle\langle k| \otimes \rho_{kk}.$$

Consider instead introducing an ancilla qubit called S, and applying $CNOT_{1\to S}$

$$\rho^{(2)} = CNOT_{1\to S}\,(\rho \otimes |0\rangle\langle 0|)\,CNOT_{1\to S}^{\dagger}.$$

$$= CNOT_{1\to S}\left(\sum_{k,k'}^{1} |k,0\rangle\langle k',0| \otimes \rho_{kk'}\right) CNOT_{1\to S}^{\dagger}.$$

$$= \sum_{k,k'}^{1} |k,k\rangle\langle k',k'| \otimes \rho_{kk'}.$$

If we ignore qubit S for the remainder, the outcome pr. can be derived from

$$\mathrm{tr}_S\, \rho^{(2)} = \sum_{k} |k\rangle\langle k| \otimes \rho_{kk}.$$

So, measurement can be replaced with unitary + ancilla which is never touched again.