List evolution rules for density matrices

Exercise Show that $\mathrm{tr}(\rho^2) = 1$ iff ρ is pure.

Pf. $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ spectral decomposition. Then

$$\rho^2 = \sum_{ij} \lambda_i \lambda_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|$$

$$= \sum_i \lambda_i^2 |\psi_i\rangle\langle\psi_i|$$

so $\mathrm{tr}(\rho^2) = \sum_i \lambda_i^2 \leq \sum \lambda_i = 1$.

equality iff each $\lambda_i \in \{0, 1\}$. But at most 1 eigenvalue of 1.

So, $\mathrm{tr}(\rho^2) = 1$ iff pure state.

Aside: For fn $f: \mathbb{R} \to \mathbb{R}$, we define

$$f(\rho) = \sum_i f(\lambda_i) |\psi_i\rangle\langle\psi_i|.$$

Why study density matrices?

① Describe prob. dist. + entanglement at the same time.

② Describe partial states of larger entangled systems.

The state of a subsystem

Alice & Bob currently are described by $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Bob goes away, never to return. How do we describe Alice's system?

Case 1: $\rho_{AB} = \rho_A \otimes \rho_B$. (unentangled) Ans: ρ_A.

Case 2: $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$. Ans: $\sum p_i \rho_i^A$.

giving rolling dice interpretation.

What about systems that are not linear combinations of unentangled?

The "state of a system" is a description from which all "experiments" can be calculated.

Operational property: Discarding Bob's system yields the same state if done before or after actions restricted to either half.

Partial Trace

$\mathcal{H}_A = \text{span}\{|a_1\rangle, \ldots |a_{d_A}\rangle\}$, $\mathcal{H}_B = \text{span}\{|b_1\rangle, \ldots |b_{d_B}\rangle\}$.

$\mathcal{H}_A \otimes \mathcal{H}_B$ has basis $\{|a_i\rangle \otimes |b_j\rangle : i \in [d_A], j \in [d_B]\}$.

Def. If X an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Then, $\text{tr}_B : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$.

$$\text{tr}_B(X) := \sum_{j=1}^{d_B} (\mathbb{1}_A \otimes \langle b_j|) X (\mathbb{1}_A \otimes |b_j\rangle).$$

Note: tr_B is a linear transformation.

Ex. $X = |a_i, b_j\rangle\langle a_k, b_\ell|$

Then $\mathrm{tr}_B(X) = \delta_{j\ell}\, |a_i\rangle\langle a_k|$. Do out in class.

Ex. $X = |EPR\rangle\langle EPR|$. Do out in class.

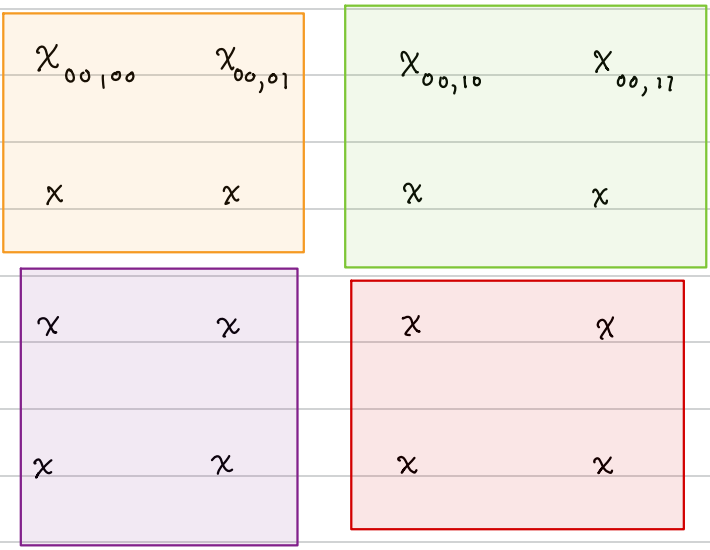
$$\mathrm{tr}_A(X) = \sum_{i=1}^{d_A} \left(\langle a_i| \otimes \mathbb{1}_B\right) X \left(|a_i\rangle \otimes \mathbb{1}_B\right)$$

Consider 2 bit system:

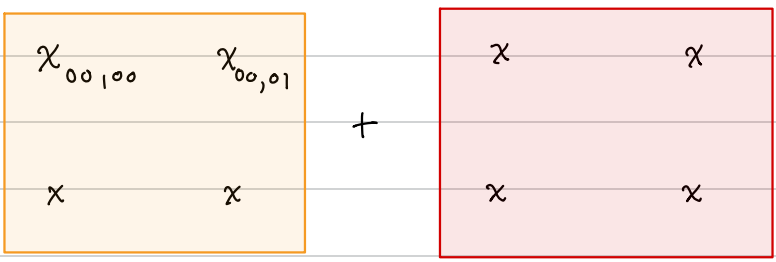
$$X = \sum_{i,j \in \{0,1\}} \sum_{k,\ell \in \{0,1\}} x_{ij,k\ell}\, |i,j\rangle\langle k,\ell|.$$

$$\mathrm{tr}_A(X) = \sum_{b,b'} \left(\sum_a x_{ab,ab'} \right) |b\rangle\langle b'|.$$

There is a nice picture.



Then, $tr_A(X) =$



$+$

Exercise Verify that $tr_B(\rho_A \otimes \rho_B) = \rho_A$.

Ex $tr_B(\rho_{AB})$ where $\rho_{AB} = \begin{bmatrix} \frac{1}{6} & \frac{-i}{6} & 0 & \frac{-1}{6} \\ \frac{i}{6} & \frac{1}{3} & 0 & \frac{1-i}{6} \\ 0 & 0 & \frac{1}{6} & 0 \\ \frac{-1}{6} & \frac{1+i}{6} & 0 & \frac{1}{3} \end{bmatrix}$.

Lem ρ_{AB} d.m. then define $\rho_A := \mathrm{tr}_B(\rho_{AB})$. Then

① ρ_A is a d.m.

② For any operator M_A on $\mathcal{H}_A$,

$$\mathrm{tr}\left(M_A \rho_A\right) = \mathrm{tr}\left(\left(M_A \otimes \mathbb{1}_B\right) \rho_{AB}\right).$$

③ For any unitaries U_A, U_B on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively,

$$\mathrm{tr}_B\left(\left(U_A \otimes U_B\right) \rho_{AB} \left(U_A \otimes U_B\right)^\dagger\right) = U_A \rho_A U_A^\dagger.$$

④ Any unrecorded measurement on Bob's system, does not change ρ_A.

⑤ For $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i \implies \rho_A = \sum_i p_i \rho_A^i$.

Pf. Start with ②,

$$\mathrm{tr}\left(M_A \mathrm{tr}_B(\rho_{AB})\right) = \mathrm{tr}\left(M_A \sum_j \left(\mathbb{1}_A \otimes \langle b_j|\right) \rho_{AB} \left(\mathbb{1}_A \otimes |b_j\rangle\right)\right)$$

$$= \mathrm{tr}\left(M_A \left(\sum_j \mathbb{1}_A \otimes |b_j\rangle\langle b_j|\right) \rho_{AB}\right) \quad \text{by cyclicity.}$$

$$= \mathrm{tr}\left((M_A \otimes \mathbb{1}_B)\, \rho_{AB} \right).$$

origin of name partial trace.

Take $M_A = \mathbb{1}$ proves $\boxed{\mathrm{tr}(\rho_A) = \mathrm{tr}(\rho_{AB}) = 1.}$

For positivity, $\langle v|\rho_A|v\rangle = \sum_j \left(\langle v| \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \langle b_j|\right) \rho_{AB} \left(\mathbb{1} \otimes |b_j\rangle\right)\left(|v\rangle \otimes \mathbb{1}\right)$

$$= \sum_j \langle v, b_j | \rho_{AB} | v, b_j \rangle$$

$$\geq 0.$$

For ③, suppose Alice & Bob apply $U_A \otimes U_B$.

$$\sigma_{AB} = (U_A \otimes U_B)\, \rho_{AB} (U_A \otimes U_B)^\dagger.$$

Calculate $\mathrm{tr}(M_A \sigma_A)$ and $\mathrm{tr}\left(M_A (U_A \rho U_A^\dagger)\right)$

$$\Rightarrow \sigma_A = U_A \rho_A U_A^\dagger.$$

Cor. Partial trace is basis independent.

Thm the partial trace is the unique linear transform T_B s.t.

$$\mathrm{tr}\left(M_A\, T_B(X)\right) = \mathrm{tr}\left((M_A \otimes \mathbb{1}_B)\, X\right).$$

Return to EPR: Partial trace proves that after tracing out half the system, each is left with a coin flip.

Maximally mixed & entangled states.

Define $\mathbb{1}/d$ and Maximally entangled, for $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$.

Generalized $|EPR\rangle$.