

One qubit axioms

- ① State is a $|\psi\rangle \in \mathbb{C}^2$
- ② Unitary transform $|\psi\rangle \mapsto U|\psi\rangle$
- ③ Measurement

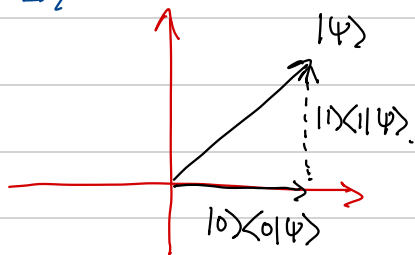
$$|\psi\rangle \mapsto \begin{cases} w \text{ pr } |\langle 0|\psi\rangle|^2, & |0\rangle \\ w \text{ pr } |\langle 1|\psi\rangle|^2, & |1\rangle. \end{cases}$$

Observe $|i\rangle\langle i|$ is the projector onto $|i\rangle$.

Ex. $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

$$\begin{aligned} |\psi\rangle &= \mathbb{1}|\psi\rangle = |0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle \\ &= (\langle 0|\psi\rangle)|0\rangle + (\langle 1|\psi\rangle)|1\rangle \end{aligned}$$



$$|\psi\rangle \mapsto \begin{cases} w \text{ pr } \|\lvert i \rangle \langle i \rvert \psi \rangle\|^2, & \frac{\lvert i \rangle \langle i \rvert \psi \rangle}{\|\lvert i \rangle \langle i \rvert \psi \rangle\|}. \end{cases}$$

Tensor spaces.

Hilbert space Vector space with inner product.

In this class $\mathcal{H} \cong \mathbb{C}^d$ for some finite d .

If $\mathcal{H}_A, \mathcal{H}_B$ are Hilbert spaces, $\mathcal{H}_A \otimes \mathcal{H}_B$ is another Hilbert space

$$\dim \mathcal{H}_A \otimes \mathcal{H}_B = \dim \mathcal{H}_A \cdot \dim \mathcal{H}_B.$$

If $\{|a_i\rangle, \dots, |a_{d_1}\rangle\}$ and $\{|b_1\rangle, \dots, |b_{d_2}\rangle\}$ are orthonormal bases then

$\{|a_i\rangle \otimes |b_j\rangle, 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$ is orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\text{Pf. } (\langle a_{i'} | \otimes \langle b_{j'} |) (|a_i\rangle \otimes |b_j\rangle) = \delta_{ii'} \delta_{jj'} \quad \square$$

Any vector $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed as

$$|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \psi_{ij} |a_i\rangle |b_j\rangle.$$

This projector based version may seem more tedious but it will be useful for multi qubits!

④ Initialization : $|0\rangle$.

Tensor product describe composite systems

qubits A in state $|\psi_A\rangle$, qubits B in state $|\psi_B\rangle$ then

$$|\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle.$$

Desiderata for axioms of multi-qubit systems

- ① If no interaction between systems, no "experiment" should differentiate just system A from the analysis of system A as half of system AB.
- ② A method for "entangling" the two systems.

Multi-qubit axioms.

$\dim 2^n$

$|x_1, \dots, x_n\rangle$

forms basis.

① n -qubit system is a unit vec in

$$\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n} \equiv \mathbb{C}^{2^n} =: \mathcal{H} \text{ Hilbert space}$$

$$|\psi\rangle = \sum_{x=0}^{2^n-1} \psi(x_1, \dots, x_n) |x_1, \dots, x_n\rangle = \sum_x \psi(x) |x\rangle$$

classical if exactly 1 value $\psi(x) = 1$.

otherwise called a "superposition".

② For any unitary $U \in \mathbb{C}^{2^n \times 2^n}$, we can transform
 $|\psi\rangle \mapsto U|\psi\rangle$

③ Measurement of the first qubit

$$\text{Let } |\psi\rangle = \alpha_0 |0\rangle |\psi_0\rangle + \alpha_1 |1\rangle |\psi_1\rangle$$

$$\text{where } \|\psi_0\rangle\| = \|\psi_1\rangle\| = 1$$

$$|\psi\rangle = \begin{bmatrix} \alpha_0 |\psi_0\rangle \\ \alpha_1 |\psi_1\rangle \end{bmatrix}$$

Measurement produces with prob. $|\alpha_i|^2$

the state $|i\rangle \otimes |\psi_i\rangle$ for $i \in \{0, 1\}$.

and outputs the classical value i .

$$|0\rangle\langle 0| \otimes \mathbb{1}_{2^{n-1}} + |1\rangle\langle 1| \otimes \mathbb{1}_{2^{n-1}}$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \mathbb{1}_{2^{n-1}} = \mathbb{1}_2 \otimes \mathbb{1}_{2^{n-1}} = \mathbb{1}_{2^n}.$$

$$|i\rangle\langle i| \otimes \mathbb{1}_{2^{n-1}} |\psi\rangle$$

$$= |i\rangle\langle i| \otimes \mathbb{1}_{2^{n-1}} \left(\sum_j \alpha_j |j\rangle |\psi_j\rangle \right)$$

$$= \sum_j \alpha_j |i\rangle\langle i|_j \otimes |\psi_j\rangle$$

$\underbrace{\qquad}_{\substack{\text{1 if } i=j \text{ else } 0.}}$

$$= \alpha_i |i\rangle \otimes |\psi_i\rangle$$

$$|\psi\rangle \mapsto \left\{ \begin{array}{l} \text{w pr } \|(|i\rangle\langle i| \otimes \mathbb{1}_{2^{n-1}}) |\psi\rangle \|^2, \quad \frac{(|i\rangle\langle i| \otimes \mathbb{1}) |\psi\rangle}{\| (|i\rangle\langle i| \otimes \mathbb{1}) |\psi\rangle \|} \end{array} \right.$$

Cor $(|i\rangle\langle i| \otimes \mathbb{1})|\psi\rangle = \sqrt{\text{pr}(i)} \cdot (\text{post-measurement state of } i).$

④ Initialization. given n -qubit $|\psi\rangle$, we can generate the state $|\psi\rangle \otimes |0\rangle$ on $(n+1)$ qubits.

Note, general transformation versus the notion of efficient transform.

Lemma Measurement of all n -qubits $\sum \alpha_x |x\rangle \mapsto |x\rangle$ w pr $|\alpha_x|^2$

Pr. Post exercise.

Def. k -qubit unitary and n -qubit unitary acting non-trivially on k -qubits if $= \mathbb{1} \otimes V$.

Does not need to act on adjacent indices.

Ex. SWAP $\text{SWAP } |a, b\rangle = |b, a\rangle.$

$$\text{SWAP}_{2,3} |x_1, x_2, x_3\rangle = |x_1, x_3, x_2\rangle \quad \text{SWAP}_{1,3} |x_1, x_2, x_3\rangle = |x_3, x_2, x_1\rangle$$

$$\text{SWAP}_{2,3} =$$

1	
0 1	
1 0	
	1
	1
	0 1
	1 0
	1

$$\text{SWAP}_{1,2} =$$

1	0
0	1
	0
	0 0 1 0
0 1 0 0	0
	0
	1
	0
	1

Joining two systems $|\psi_A\rangle$ and $|\phi_B\rangle$ becomes

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\phi_B\rangle.$$

We have simply defined two systems "adjacent" to each other.
We shouldn't be able to distinguish an experiment on just A
from an experiment on only A while considering both A and B.

① Unitary transformation: U_A .

Apply U_A , then join with B.

$$|\psi_A\rangle \mapsto U_A |\psi_A\rangle \mapsto U_A |\psi_A\rangle \otimes |\phi_B\rangle$$

Join with B, then apply U_A

$$\begin{aligned} |\psi_A\rangle &\mapsto |\psi_A\rangle \otimes |\phi_B\rangle \mapsto (U_A \otimes \mathbb{1}_B) |\psi_A\rangle \otimes |\phi_B\rangle \\ &= U_A |\psi_A\rangle \otimes |\phi_B\rangle. \end{aligned}$$

② Measurement (for simplicity assume $|\psi\rangle$ is 1 qubit)

$$|\psi_A\rangle = \alpha |0\rangle + \alpha' |1\rangle$$

Measure then join:

$$|\psi_A\rangle \mapsto |b\rangle \text{ w pr. } |\alpha_b|^2 \mapsto |b\rangle \otimes |\phi_B\rangle \text{ w pr. } |\alpha_b|^2$$

Join them measure:

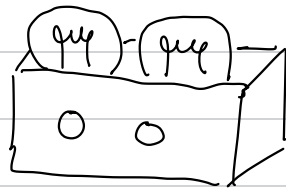
$$|\psi_A\rangle \mapsto |\psi_A\rangle \otimes |\phi_B\rangle = \alpha_0 |0\rangle |\phi_B\rangle + \alpha_1 |1\rangle |\phi_B\rangle$$

$$\mapsto |b\rangle |\phi_B\rangle \text{ w pr } |\alpha_b|^2.$$

$\Rightarrow$ Considering two non-interacting systems does not change the experimental outcomes of studying one of the systems.

Deutsch-Jozsa game

Box with 2 buttons & 2 lights.



Described by fn $f: \{0,1\} \rightarrow \{0,1\}$.

Classically, 2 queries to decide if $f(0) = f(1)$.

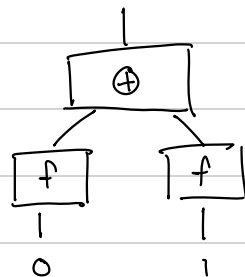
Quantum, only 1 query necessary.*1

- We won't learn value of $f(0)$ or $f(1)$.

- Quantum algs can access some "orthogonal" info.

(Aside: in this case, the info is in the Fourier basis.)

Classical algorithmic for $f(0) \stackrel{?}{=} f(1)$:



What does quantum access to f mean?

Attempt 1: apply transform $\begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} |x\rangle \mapsto |f(x)\rangle$

Why does this not work? Ans: not necessarily unitary

Attempt 2: what about $\begin{pmatrix} (-1)^{f(0)} & 0 \\ 0 & (-1)^{f(1)} \end{pmatrix} (= : V_f)$

$|x\rangle \mapsto (-1)^{f(x)} |x\rangle$ Unitary ✓

Issue: Is this actually physically realistic?

Attempt 3: Reversible version of f :

$U_f: (x, y) \mapsto (x, y \oplus f(x)).$