

Example of q. information: The Bomb Tester

Def Single-qubit basis measurement. $\{|b_0\rangle, |b_1\rangle\}$ basis.

$$\text{goal } |\psi\rangle \mapsto \begin{cases} w & \text{pr } |\langle b_0 | \psi \rangle|^2, |b_0\rangle \\ w & \text{pr } |\langle b_1 | \psi \rangle|^2, |b_1\rangle \end{cases}$$

Lem Implementable from current axioms.

Pf. $U = |b_0\rangle\langle 0| + |b_1\rangle\langle 1|$ $U = \begin{pmatrix} | & | \\ |b_0\rangle & |b_1\rangle \\ | & | \end{pmatrix}$

U unitary as orthonormal cols, $U^\dagger$ also unitary.

$$U^\dagger = |0\rangle\langle b_0| + |1\rangle\langle b_1| = \sum_i |i\rangle\langle b_i|$$

Alg on input $|\psi\rangle \in \mathbb{C}^2$:

- ① Apply $U^\dagger$
- ② Standard-basis measurement
- ③ Apply U .

After ①, state is $U^\dagger |\psi\rangle$

After ②, w pr $\underbrace{|\langle i | U^\dagger |\psi\rangle|^2}_{= |\langle b_i | \psi \rangle|^2}$, $|i\rangle$ for $i \in \{0, 1\}$

After ③, w pr $|\langle i | U^\dagger |\psi\rangle|^2$, $\underbrace{U|i\rangle}_{= |b_i\rangle}$

So, any single-qubit basis measurement is implementable.

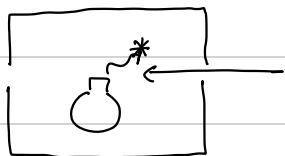
Bomb-tester thought experiment (Elitzur-Vaidman 1980)

There's a purported bomb but possibly a dud.



dud

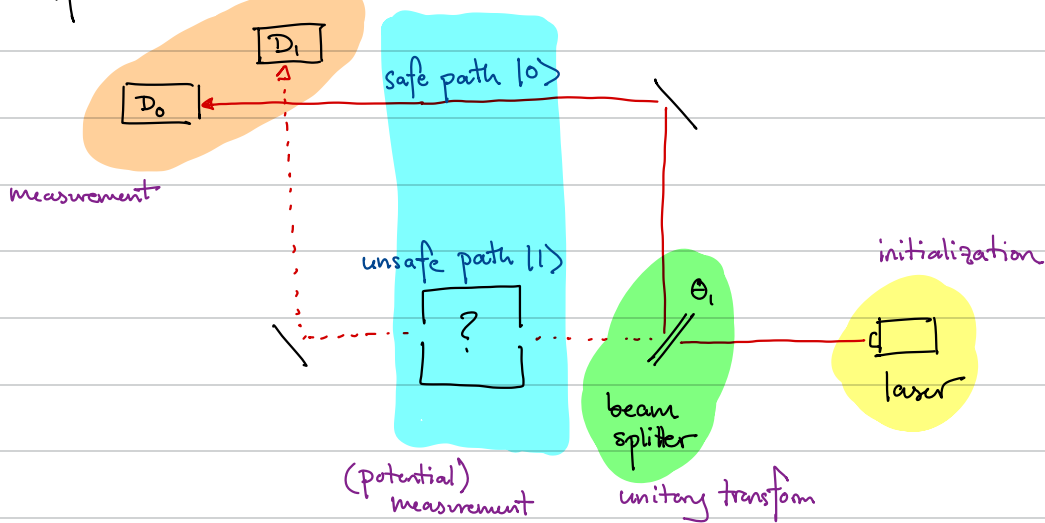
photon passes through
with no effect



photon enters but then everything
explodes

Can we figure out if bombs or not while minimizing the chance of explosion?

A quantum solution



θ_1 tunes the superposition, i.e. acts like a rotation unitary.

$$R_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \text{After beam-split state is}$$

$$R_{\theta_1} |0\rangle = \cos \theta_1 |0\rangle + \sin \theta_1 |1\rangle.$$

Case 1: It's a dud.

$$\Pr[\text{accept} \mid \text{dud}] = \cos^2 \theta_1$$

$$\Pr[\text{reject} \mid \text{dud}] = \sin^2 \theta_1$$

$$\Pr[\text{explode} \mid \text{dud}] = 0$$

Question for class:

Is there anything quantum going on?

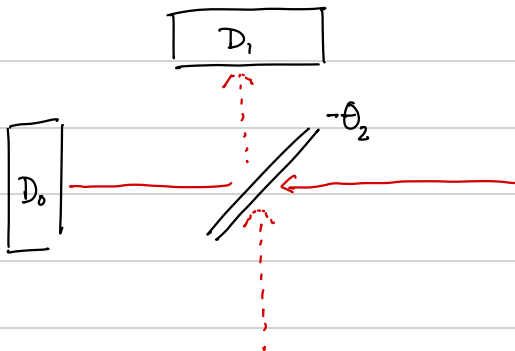
Case 2: It's a bomb.

$$\Pr[\text{accept} \mid \text{bomb}] = \cos^2 \theta_1$$

$$\Pr[\text{reject} \mid \text{bomb}] = 0$$

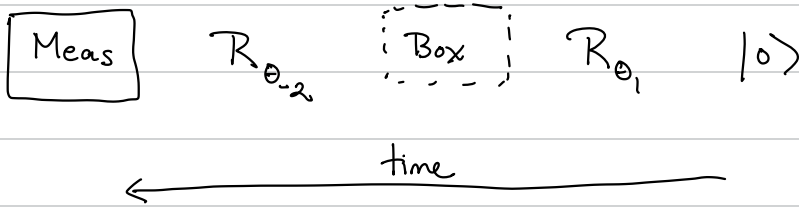
$$\Pr[\text{explode} \mid \text{bomb}] = \sin^2 \theta_1$$

Small θ_1 ensures low risk of explosion. Moreover a notification of "reject" is the all-clear. But this test is effectively classical. Can we do better?



Second beam-splitter

Now what's going on?



If dud, state being measured is

$$\cos(\theta_1 - \theta_2) |0\rangle + \sin(\theta_1 - \theta_2) |1\rangle.$$

Then, $\Pr[\text{accept} | \text{dud}] = \cos^2(\theta_1 - \theta_2)$

$$\Pr[\text{reject} | \text{dud}] = \sin^2(\theta_1 - \theta_2)$$

$$\Pr[\text{explode} | \text{dud}] = 0.$$

If bomb, $\Pr[\text{explode} | \text{bomb}] = \sin^2 \theta_1.$

If it doesn't explode, collapse to $|0\rangle$ (safe path).

So second measurement is of $|- \theta_2\rangle = \cos(-\theta_2) |0\rangle + \sin(-\theta_2) |1\rangle.$

By conditional probability,

$$\Pr(\text{accept} \mid \text{bomb}) = \cos^2 \theta_1 \cos^2 \theta_2$$

$$\Pr(\text{reject} \mid \text{bomb}) = \cos^2 \theta_1 \sin^2 \theta_2$$

$$\Pr(\text{explode} \mid \text{bomb}) = \sin^2 \theta_1.$$

Choose $\sin^2 \theta_1 = \epsilon$, explosion risk. In order to force only 1 outcome with a dud, pick $\theta_1 = \theta_2$.

Then dud always accepts and

$$\Pr(\text{accept} \mid \text{bomb}) = (1 - \epsilon)^2 = 1 - 2\epsilon + \epsilon^2$$

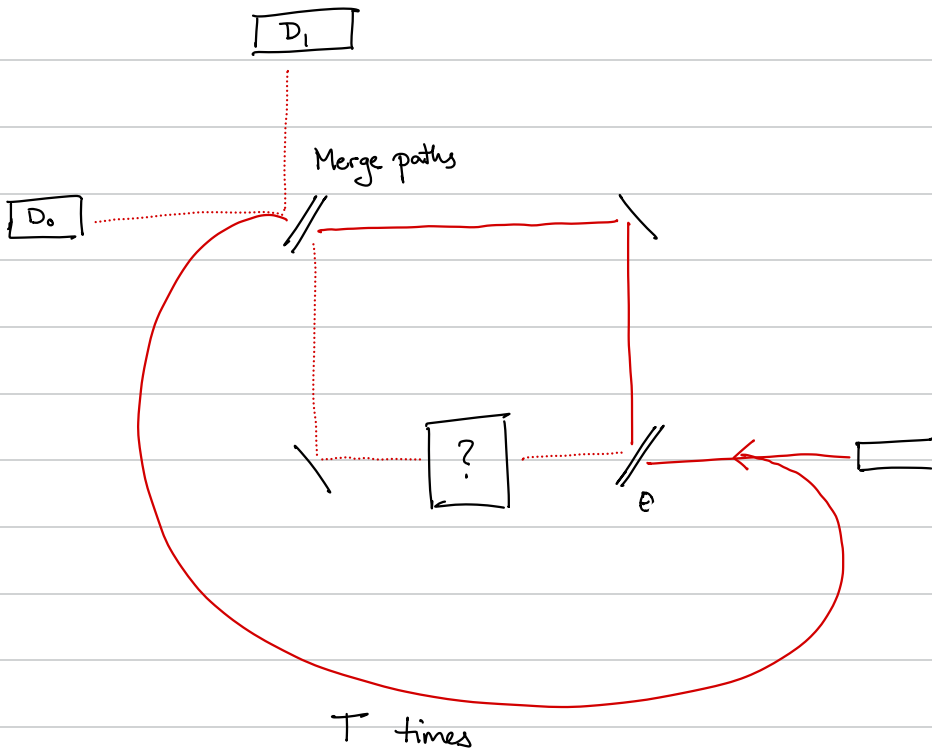
$$\Pr(\text{reject} \mid \text{bomb}) = (1 - \epsilon)\epsilon = \epsilon - \epsilon^2$$

$$\Pr(\text{explode} \mid \text{bomb}) = \epsilon = \epsilon$$

Certified the presence of a bomb without any explosion!

If "reject" is seen, bomb is known. Trouble is detection prob $<$ detonation probability. How do we boost this to something that detects with high prob but also explodes with the same prob.

Recycled photon bomb-test



Let's try to build on the interference patterns and instead of using a fresh photon each time, let's recycle the photon.

T times let's pass it through the beam splitter and after that let's measure with the detectors.

Case 1: Dud.

Each pass applies R_θ for a total of $(R_\theta)^T = R_{T\theta}$.

So measured state is $\cos T\theta |0\rangle + \sin T\theta |1\rangle$

$\Rightarrow \cos^2 T\theta$ of accept, $\sin^2 T\theta$ of reject.

Case 2: Bomb.

Claim Conditioned on surviving past rounds, starting state is $|0\rangle$.

Then after rotation, $\cos^2 \theta |0\rangle + \sin^2 \theta |1\rangle$.

with prob of surviving in round t , conditioned on surviving first $t-1$ rounds is $\cos^2 \theta$.

$$\begin{aligned} \text{Prob of surviving} &= (\cos^2 \theta)^T \geq 1 - T(1 - \cos^2 \theta) \\ &= 1 - T \sin^2 \theta. \end{aligned}$$

$$\text{So } \Pr[\text{explode} | \text{bomb}] = T \sin^2 \theta \leq T \theta^2.$$

If it doesn't explode, state is 0.

Pick $\Theta = \frac{\pi}{2T}$. Then,

$$\Pr[\text{reject} \mid \text{dud}] = \sin^2 T\Theta = 1.$$

$$\Pr[\text{explode} \mid \text{bomb}] = \frac{\pi^2}{4T}$$

$$\Pr[\text{accept} \mid \text{bomb}] = 1 - \frac{\pi^2}{4T}.$$

If we tolerate risk ϵ , pick $T = \Omega\left(\frac{1}{\epsilon}\right)$.

Optimal demonstration of interaction-free measurement.

Multiple qubits

$$A = \begin{pmatrix} A_{11} & \dots & A_{1n_1} \\ \vdots & & \vdots \\ A_{m_1 n_1} & & A_{m_1 n_1} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{1n_2} \\ \vdots & \vdots \\ B_{m_2 n_2} & B_{m_2 n_2} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} A_{11} \cdot B & \dots & \\ \vdots & & \vdots \\ A_{m_1 n_1} B & & \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} & \dots & \\ \vdots & & \vdots \\ A_{m_1 n_1} \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} & & \end{pmatrix}$$

Properties: $(A \otimes B)(C \otimes D) = AC \otimes BD$ assuming dims match

$$(A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \text{ (when invertible)}$$

$$A \otimes \left(\sum b_i B_i \right) = \sum b_i (A \otimes B_i)$$

$$\left(\sum a_i A_i \right) \otimes B = \sum a_i (A_i \otimes B).$$

Tensor products of vectors as vectors are 1-D matrices

Exercises $|0\rangle \otimes |1\rangle \otimes |0\rangle$ and $|0\rangle \otimes |1\rangle \otimes |1\rangle$

Notation $|0\rangle \otimes |1\rangle \otimes |0\rangle = |0\rangle|1\rangle|0\rangle = |0, 1, 0\rangle.$

Rule $|x_1, \dots, x_n\rangle$ for $x_i \in \{0, 1\}$ is the x^{th} basis vec. when x interpreted in binary

Often we just ignore commas and write as $|x\rangle.$

$\{|x\rangle : x \in \{0, 1\}^n\}$ form computational basis of $(\mathbb{C}^2)^{\otimes n}.$