CSE 534 Autumn 2025: Set 0

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Due date: None!

> This problem set is designed to ensure that you are up to speed on prerequisite mathematics and is neither graded nor required. However, all students enrolled are expected to be able to answer questions of this level with only moderate difficulty. You *should* ask the TA if you are confused about any concept on this problem set.

Problem 1 (Polar Representations)**.** Consider a complex number $z \in \mathbb{C}$ such that $z = re^{i\theta}$ for $r \in \mathbb{R}^{\geq 0}$ and $\theta \in [0, 2\pi)$.

1. Show that every complex number $a + bi$ for $a, b \in \mathbb{R}$ can be expressed with a polar representation. Show that the polar representation is unique for $r > 0$; that is $z = r'e^{i\theta'}$ if and only if $r' = r$ and $\theta' = \theta$.
2. For $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$, what is the polar representation of the complex conjugate z^*? What is the polar representation of $z \cdot z'$?
3. The multiplicative inverse, denoted z^{-1}, is the unique complex number such that $z \cdot z' = 1$. What is the multiplicative inverse of $z = re^{i\theta}$, and what is the multiplicative inverse of $z = a + bi$?
4. For any natural number $n \in \mathbb{N}$, calculate the n roots (including multiplicities) of $re^{i\theta}$ and express them in terms of $\omega_n = e^{2\pi i/n}$.

Problem 2 (Bra-ket notation)**.** Bra-ket notation is a quantum mechanics notation that is popularly used in quantum information and computation. We use $|v\rangle$ to denote a (column)-vector in $\mathbb{C}^d$. Recall that a vector is a matrix with only one column. Equivalently $|v\rangle \in \mathbb{C}^{d\times 1}$. We use $\langle v| \in \mathbb{C}^{1\times d}$ to denote the row-vector with complex conjugate entries as that of $|v\rangle$. So if

$$|v\rangle = \begin{pmatrix} v(0) \\ v(1) \\ \vdots \\ v_(d-1) \end{pmatrix}, \text{ then } \quad \langle v| = \begin{pmatrix} v(0)^* & v(1)^* & \cdots & v(d-1)^* \end{pmatrix}, \tag{1}$$

in terms of the entry-wise values of $|v\rangle$.

1. We use $\langle v|v\rangle$ as a short-form for $\langle v| \cdot |v\rangle$. What is the value of $\langle v|v\rangle$? What is this quantity in traditional mathematical terms? Show that this value is always real.

2. What is the matrix representation of $|v\rangle\langle v|$? What is the trace of this matrix? How could you have calculated that using the cyclicity of the trace?

3. We use $\||v\rangle\|$ to denote $\sqrt{\langle v|v\rangle}$. A vector is unit-norm if its norm equals 1. For notation, we use the following convention to describe the unit vectors pointing in each of the coordinate directions:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{2}$$

   These vectors form a basis for $\mathbb{C}^d$ which is known as the *standard* or *canonical* basis. How can we express the vector $|v\rangle$ in terms of these basis vectors?

4. Similarly, we use $|i\rangle\langle i|$ as a short-form for $|i\rangle\cdot\langle i|$. What is the matrix $\sum_{i=0}^{d-1} |i\rangle\langle i|$ in traditional mathematical terms?

5. For $i \in \{0, \dots, d-1\}$, what is the value of $\langle i|v\rangle$ for $|v\rangle$ from before? Show that

$$|v\rangle = \sum_{i=0}^{d-1} \langle i|v\rangle\, |i\rangle\,. \tag{3}$$

   Hint: Use the decomposition of $\mathbb{I}$ from the previous question and use that $\langle i|v\rangle$ is a scalar.

6. Let $|v\rangle$ and $|w\rangle$ be unit vectors. Let $M = |w\rangle\langle v|$. What is the value of $M\,|v\rangle$? Let $|v^\perp\rangle$ be a vector such that $\langle v|v^\perp\rangle = 0$. What is $M\,|v^\perp\rangle$?

7. Let M be a square matrix $\in \mathbb{C}^{d\times d}$ with columns $|v_0\rangle, |v_1\rangle, \dots, |v_{d-1}\rangle$. Show that

$$M = \sum_{i=0}^{d-1} |v_i\rangle\langle i|\,. \tag{4}$$

8. Let M be a matrix in $\mathbb{C}^{d\times d}$ and $|v\rangle$ a vector in $\mathbb{C}^d$. Argue that the $\langle v|\,M^\dagger$ is the conjugate transpose of $M\,|v\rangle$.

If that went well, you should be able to start using bra-ket notation. It will become more natural over time!

Problem 3 (Bases for vector spaces)**.** A set $V \subset \mathbb{C}^d$ is a subspace if it is closed under scalar multiplication and addition. A basis B for a vector space is a collection of vectors such that every vector can be expressed as a linear combination of vectors from B and B is of minimal cardinality.

1. Show that all bases for a vector space $V \subset \mathbb{C}^d$ have the same cardinality, which is an integer between 0 and d.

2. Show that $|v_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{C}^2$. Express the vectors $|0\rangle$ and $|1\rangle$ (see definitions above) in terms of this basis. The vectors $|v_0\rangle$ and $|v_1\rangle$ are used so much in quantum computation that they have special names: $|+\rangle$ and $|-\rangle$.

3. Express the vectors $|0\rangle$ and $|1\rangle$ in terms of $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

A basis is called orthonormal if all the vectors in the basis are orthogonal to each other and of unit norm. To define orthogonality and norm, we will need the next problem:

Problem 4 (Inner-product spaces)**.** Consider a vector space V with a function $f : V \times V \to \mathbb{C}$ satisfying the following properties: For all vectors $u, v, w \in V$,

- (linearity) for all $\alpha, \beta \in \mathbb{C}$, $f(u, \alpha v + \beta w) = \alpha f(u, v) + \beta f(u, w)$.
- (complex conjugate symmetry) $f(u, v) = f(v, u)^*$.
- (positivity) $f(u, u) \in \mathbb{R}^+$ and $f(u, u) = 0$ iff $u = 0$.

Such a vector space is called an inner product space. When considering finite-dimensional vector spaces, inner product spaces are *equivalent* to Hilbert spaces.

1. Show that $f(\alpha u, v) = \alpha^* f(u, v)$.

2. Show that the bilinear form given by $f(|u\rangle, |v\rangle) = \langle u|v\rangle$ for $|u\rangle, |v\rangle \in \mathbb{C}^d$ makes $\mathbb{C}^d$ an inner product space.

3. In a lot of mathematical notation, the function f is often expressed as a bilinear form: $\langle \cdot, \cdot \rangle \overset{\text{def}}{=} f(\cdot, \cdot)$. The bra-ket notation was adopted in part because it matches this. We previously defined the norm as $\||v\rangle\| = \sqrt{\langle v|v\rangle}$. Show that the function $\|\cdot\| : \mathbb{C}^d \to \mathbb{R}^+$ defined from the inner product yields a metric which is a function satisfying linearity, positivity, and the triangle inequality.

4. Recall that in a metric space, by the Pythagorean theorem, two unit vectors $|u\rangle$ and $|v\rangle$ are orthogonal if $\||u\rangle - |v\rangle\| = \sqrt{2}$. Show that this is true iff $\langle u|v\rangle = 0$.

5. Generalize to show that if $|\langle u|v\rangle| \leq \epsilon$, then

$$\||u\rangle - |v\rangle\|^2 \geq 2(1 - \epsilon). \tag{5}$$

   This says that when the inner product is small, then the two vectors are far from each other.

6. Show the converse is false. Find vectors such that $|\langle u|v\rangle| = 1$ but they are maximally far apart.
7. Show that if $|\langle u|v\rangle| \geq 1 - \epsilon$, then there exists an angle $\theta \in [0, 2\pi)$ such that

$$\left\| |u\rangle - e^{i\theta} |v\rangle \right\| \leq \sqrt{2\epsilon}. \tag{6}$$

This shows that a large inner product implies that the two vectors are close "up to phase".

Problem 5 (Linear transformations)**.** Recall/show that over a vector space $V \subseteq \mathbb{C}^d$, any linear transformation $T : V \to V$ has a matrix representation such that $T(|v\rangle) = M_T |v\rangle$. For this reason, we often use the same symbol to represent the linear transform and its matrix representation. For a bilinear form $\langle \cdot, \cdot \rangle$ defining an inner product space, the adjoint of a linear transform T is the transformation $T^\dagger$ such that

$$\langle T^\dagger u, v\rangle = \langle u, Tv\rangle. \tag{7}$$

1. Show that the adjoint of a linear transform with matrix representation T for the bilinear form $\langle \cdot|\cdot\rangle$ is defined by the conjugate transpose.
2. A matrix $H \in \mathbb{C}^{d\times d}$ is Hermitian if $H^\dagger = H$ where $H^\dagger$ is the conjugate transpose.
   (a) Show that the diagonal coordinates of H must be real.
   (b) Show that the eigenvalues of H must be real.
   Hint: assume $|v\rangle$ is a unit eigenvector of H of eigenvalue $\lambda \in \mathbb{C}$. Argue that then $\lambda = \lambda^*$.
   (c) Let $|v\rangle$ be a unit eigenvector of eigenvalue $\lambda \in \mathbb{R}$. Let $H' = H - \lambda |v\rangle\langle v|$. Show that $H' |v\rangle = 0$. Next, let $|w\rangle$ be a vector orthogonal to $|v\rangle$. Show that $H' |w\rangle = H |w\rangle$.
   (d) Let $\Pi = \mathbb{I} - |\lambda\rangle\langle\lambda|$. Recall that a matrix is a projection if it equals its square, or equivalently, its eigenvalues are either 1 or 0. Show that Π is a projection — i.e. $\Pi^2 = \Pi$. Show that $H' = \Pi H \Pi$.
   (e) Use this and the Gram-Schmidt process to show that there exist eigenvalues $\lambda_0, \ldots, \lambda_{d-1} \in \mathbb{R}$ and an orthonormal set of eigenvectors $|v_0\rangle, \ldots, |v_{d-1}\rangle$ such that

$$H = \sum_{i=0}^{d-1} \lambda_i |v_i\rangle\langle v_i|. \tag{8}$$

3. A matrix $U \in \mathbb{C}^{d\times d}$ is unitary if $U^\dagger U = \mathbb{I}$. Show that the following are equivalent by proving that each implies the next and the last implies the first.
   - U is unitary.
   - U maps an orthonormal basis to an orthonormal basis.
     Hint: If $|u\rangle$ and $|v\rangle$ are orthogonal, then the bilinear form $\langle u|v\rangle = 0$.
     What is the bilinear form between $U|u\rangle$ and $U|v\rangle$?

- U maps any unit vector to another unit vector.