Lecture 6

Oct 13, 2025

Is $\cos^2 \pi/8$, the optimal quantum strategy?

Can we do better if Alice and Bob share multiple EPR pairs?

Answer: <u>YES</u>!

But to prove it, we will need a way to characterize all quantum strategies Alice & Bob could apply.

General quantum strategy of Alice

Input: ρ_{AB} ← some state over Alice and Bob.
Could include ancillas.

Her algorithm:

- Apply unitary $\mathcal{U}_1$ to <u>her</u> qubits.
- Measure qubit 1.
- Apply $\mathcal{U}_2$ to <u>her</u> qubits.
- Measure qubit 1.

⋮

- Apply $\mathcal{U}_T$ to her qubits.
- Measure qubit 1 and output value.

Note: this is general enough to include any classical processing of measurements by her 1 result.

We need to simplify her algorithm.

Step 1: Principle of deferred measurement

Alice's strategy is equivalent to a strategy with only 1 measurement (the final measurement). The new strategy uses T additional ancilla qubits.

Pf.

Recall CNOT $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ or CNOT $|x\rangle|y\rangle = |x\rangle|y\oplus x\rangle$.

Let σ_{AB} be the state of the computation before the measurement of the 1^{st} qubit. After the measurement the state will be:

$$\sigma_{AB}' = \sum_{k\in\{0,1\}} (|k\rangle\langle k|\otimes \mathbb{1})\,\sigma_{AB}\,(|k\rangle\langle k|\otimes \mathbb{1}).$$

Consider instead initializing an additional ancilla so the state is $\sigma_{AB}\otimes|0\rangle\langle 0|_{Anc}$. Apply CNOT between qubit 1 and ancilla. Then ignore the ancilla for the remainder of the algorithm.

Since the remainder of the alg. ignores the ancilla, we know the measurements are equivalent to that of the state

$$\mathrm{tr}_{anc}\left(\mathrm{CNOT}_{1,anc} \left(\sigma_{AB} \otimes |0\rangle\langle 0|_{anc} \right) \mathrm{CNOT}_{1,anc} \right)$$

Suffices to show that this equals σ'_{AB}.

Let $\sigma_{AB} = \sum_{i,j \in \{0,1\}} |i\rangle\langle j|_1 \otimes \sigma_{ij}$ ← generic decomp.

$$\left(\sigma_{ij} = (\langle i| \otimes \mathbb{1}) \sigma (|j\rangle \otimes \mathbb{1}) \right) \quad \sigma = \begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix}.$$

$$\mathrm{tr}_{anc}\left(\mathrm{CNOT}_{1,anc} \left(\sigma_{AB} \otimes |0\rangle\langle 0|_{anc} \right) \mathrm{CNOT}_{1,anc} \right)$$

$$= \mathrm{tr}_{anc}\left(\sum_{i,j \in \{0,1\}} |i,i\rangle\langle j,j|_{1,anc} \otimes \sigma_{ij} \right)$$

$$= \sum_{k \in \{0,1\}} \langle k|_{anc} \otimes \mathbb{1}_{AB} \left(\sum_{i,j \in \{0,1\}} |i,i\rangle\langle j,j|_{1,anc} \otimes \sigma_{ij} \right) |k\rangle_{anc} \otimes \mathbb{1}_{AB}$$

equals 0 unless $k = i$ and $j = k$

$$= \sum_{k \in \{0,1\}} |k\rangle\langle k|_1 \otimes \sigma_{kk}$$

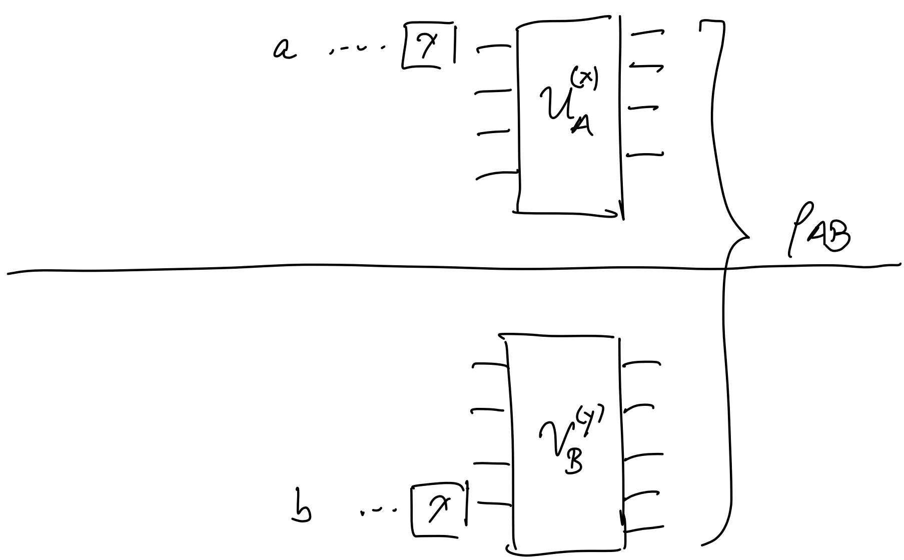
Notice $\sigma' = \sum_{k\in\{0,1\}} (|k\rangle\langle k| \otimes \mathbb{1}) \sigma_{AB} (|k\rangle\langle k| \otimes \mathbb{1})$.

$$= \sum_k |k\rangle\langle k| \otimes \sigma_{kk}.$$

Therefore, entangling with ancilla and "discarding" ancilla is equivalent to measurement.

So, Alice strategy is to apply a unitary U_A and then measure the first qubit, without loss of generality.

Same with Bob.



What is Alice's prob of outputting a ?

$$\Pr(\text{outputting } a) = \operatorname{tr}\left(\left(|a\rangle\langle a| \otimes \mathbb{1} \right) U_A^{(x)} \rho_A U_A^{(x)\dagger} \right)$$

$$= \operatorname{tr}\left(U_A^{(x)\dagger} \left(|a\rangle\langle a| \otimes \mathbb{1} \right) U_A^{(x)} \rho_A \right)$$

Let $A_x^{(a)} = U_A^{(x)\dagger} \, |a\rangle\langle a| \otimes \mathbb{1} \, U_A^{(x)}$.

Then $\Pr(\text{outputting } a) = \operatorname{tr}\left(A_x^{(a)} \rho \right)$.

Notice $A_x^{(0)} + A_x^{(1)} = U_A^{(x)\dagger} \left(\sum_a |a\rangle\langle a| \otimes \mathbb{1} \right) U_A^{(x)}$

$$= U_A^\dagger \mathbb{1} U_A = \mathbb{1}$$

and $A_x^{(a)} \geq 0$, and $(A_x^{(a)})^2 = A_x^{(a)}$ projector.

This is a special case of a

pos. valued operator valued measurement.

will show up in hw 2.

For now, suffices to observe that Alices strategy can be described by $A_x^{(0)}, A_x^{(1)}$ proj. for $A_x^{(0)} + A_x^{(1)} = \mathbb{1}_A$.

In general, for the CHSH game, Alice's action depends on input x, so her strategy can be expressed by pairs

$\{A_x^0, A_x^1\}_{x \in \{0,1\}}$ s.t. A_x^a proj. and $A_x^0 + A_x^1 = \mathbb{1}_A$.

Same with Bob: $\{B_y^0, B_y^1\}$

$$\text{Then } \Pr\left[a,b \mid x,y\right] = \operatorname{tr}\left(A_x^{(a)} \otimes B_y^{(b)} \rho_{AB}\right).$$

Conceptual switch: Have Alice & Bob answer with ± 1 instead. Then win condition $a \oplus b = xy$

becomes $a \cdot b = (-1)^{xy}$. Just mathematical.

$$\Pr[\text{win}] = \sum_{x,y} \Pr[x,y] \cdot \sum_{a,b} \Pr[a,b \mid x,y] \cdot \mathbb{1}\{a \cdot b = (-1)^{xy}\}$$

$$= \frac{1}{4} \sum_{a,b,x,y}^{1} \mathbb{1}\{a \cdot b = (-1)^{xy}\} \operatorname{tr}\left(A_x^{(a)} \otimes B_y^{(b)} \rho_{AB}\right)$$

$$= \frac{1}{4} \sum_{a,b,x,y} \left(\frac{1}{2} + \frac{a \cdot b \cdot (-1)^{xy}}{2} \right) tr\left(A_x^{(a)} \otimes B_y^{(b)} \rho_{AB} \right)$$

$$= \frac{1}{2} + \frac{1}{8} \sum_{a,b,x,y} ab(-1)^{xy} \; tr\left(A_x^{(a)} \otimes B_y^{(b)} \rho_{AB} \right)$$

$$(*) \quad = \frac{1}{2} + \frac{1}{8} tr\left(\left(\sum_{a,b,x,y} (-1)^{xy} (a A_x^{(a)} \otimes b B_y^{(b)}) \right) \rho_{AB} \right).$$

To simplify, lets define $A_x := A_x^{(1)} - A_x^{(-1)}$.

Since $A_x^{(1)} + A_x^{(-1)} = \mathbb{1}_A$ and $A_x^{(a)}$ are projectors,

$$A_x = \mathbb{1}_A - 2 A_x^{(1)} \quad \text{and}$$

$$A_x^2 = \mathbb{1}_A + 4 A_x^{(1)\,2} - 4 A_x^{(1)} = \mathbb{1}_A \quad \text{since } A_x^{(1)} \text{ proj.}$$

so A_x^2 has eigenvalues $(-1, 1)$.

Such matrices are called "binary observables".

Back to solving $(*)$. Notice that

$$A_x \otimes B_y = \left(\sum_a a A_x^{(a)} \right) \otimes \left(\sum_b b B_y^{(b)} \right)$$

$$= \sum_{a,b} a A_x^{(a)} \otimes b B_y^{(b)}.$$

So $(*) = \frac{1}{2} + \frac{1}{8} \operatorname{tr}\Bigg(\underbrace{\Big(\sum_{x,y} (-1)^{xy} A_x \otimes B_y \Big)}_{:= \mathrm{CHSH}} \rho_{AB} \Bigg)$

CHSH is an operator $\in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. So

Claim: $\operatorname{tr}(\mathrm{CHSH}\ \rho_{AB}) \leq \underbrace{\|\mathrm{CHSH}\|_{op}}_{\text{max singular value.}}$

Pf. $\rho_{AB} = \sum p_i |\psi_i\rangle\langle\psi_i|$.

unit norm

Then $\operatorname{tr}(\mathrm{CHSH}\ \rho_{AB}) = \sum_i p_i \langle\psi_i|\mathrm{CHSH}|\psi_i\rangle$

$\leq \sum_i p_i \|\mathrm{CHSH}\| = \|\mathrm{CHSH}\|$. □

So, $\Pr[\text{win}] \leq \frac{1}{2} + \frac{1}{8}\|\mathrm{CHSH}\|$.

Note: $\cos^2 \pi/8 = \frac{1}{2} + \frac{2\sqrt{2}}{8}$.

So suffices to prove, $\|\mathrm{CHSH}\| \leq 2\sqrt{2}$, or $\|\mathrm{CHSH}^2\| \leq 8$.

Pf.

$$\mathrm{CHSH} = A_0 \otimes B_0 + A_1 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_1 .$$

$$= (A_0 + A_1) \otimes B_0 + (A_0 - A_1) \otimes B_1 ,$$

Use $A_0^2 = A_1^2 = B_0^2 = B_1^2 = \mathbb{1}$ to get

$$\mathrm{CHSH}^2 = (A_0 + A_1)^2 \otimes \mathbb{1} + (A_0 - A_1)^2 \otimes \mathbb{1}$$ (no commutation!)

$$+ (A_0 + A_1)(A_0 - A_1) \otimes B_0 B_1$$

$$+ (A_0 - A_1)(A_0 + A_1) \otimes B_1 B_0 .$$

$$= 4\mathbb{1} + (\underbrace{A_0 A_1 + A_1 A_0 - A_0 A_1 - A_1 A_0}_{0}) \otimes \mathbb{1}$$

$$+ (A_0 A_1 - A_1 A_0) \otimes (-B_0 B_1)$$

$$+ (A_0 A_1 - A_1 A_0) \otimes (B_1 B_0)$$

$$= 4\mathbb{1} + [A_0, A_1] \otimes [B_1, B_0]$$

where $[A_0, A_1]$ = commutator $:= A_0 A_1 - A_1 A_0$.

Now $\|[A_0,A_1]\| \le \|A_0A_1\| + \|A_1A_0\|$

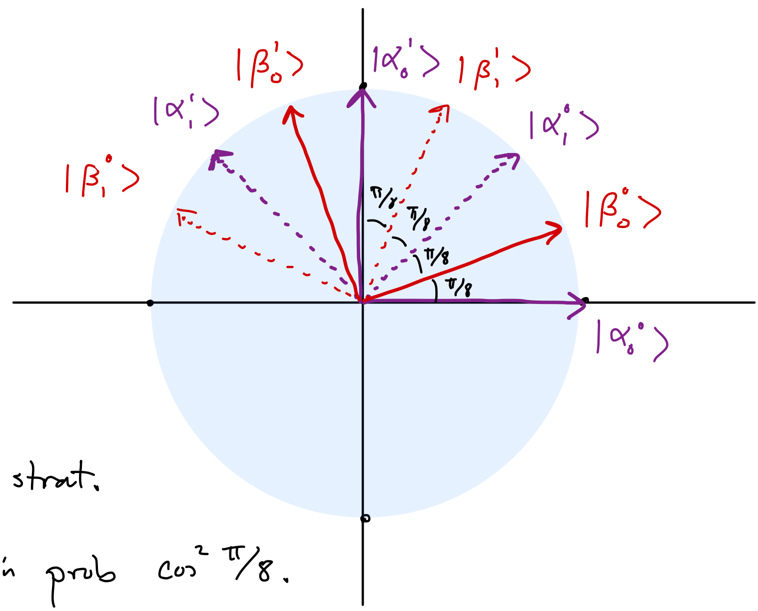
$$\le \|A_0\|\cdot\|A_1\| + \|A_1\|\cdot\|A_0\| = 2.$$

So $\|CHSH\|^2 = 4 + \|[A_0,A_1]\|\cdot\|[B_0,B_1]\|$

$$\le 4 + 2\cdot 2 = 8. \quad \blacksquare$$

What have we solved?

We proved, the following strategy wins with prob. $\cos^2 \pi/8$.



And that every q. strat. is bounded by win prob $\cos^2 \pi/8$.

Def. (Value of game) For game G,

$\omega(G)$ = max prob winning over classical strat

$\omega^*(G)$ = sup prob. winning over q. strat.

$\omega^*(\text{CHSH}) = \cos^2 \pi/8$ and $\omega(\text{CHSH}) = 3/4$.

What was the opt strategy, we found?

$A_0^* = |\alpha_0^0\rangle\langle\alpha_0^0| - |\alpha_0^1\rangle\langle\alpha_0^1| = Z$ (phase-flip)

$A_1^* = X$ (bit-flip)

$B_0^* = H$ (Hadamard)

$B_1^* = \tilde{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$ ("rotated" Hadamard)

Lets notice that $A_0^* A_1^* = -A_1^* A_0^*$ and $B_0^* B_1^* = -B_1^* B_0^*$.

(anti-commutation).

Furthermore, $|EPR\rangle$ is the unique $2\sqrt{2}$ eigenvector of

$$\text{CHSH}^* = A_0 \otimes B_0 + A_1 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_1$$

(easy to computer verify).

Are there any other optimal strategies?

Thm All optimal strategies are unitarily equivalent to to the strategy $(A_0^*, A_1^*, B_0^*, B_1^*, |EPR\rangle)$ given.

we will formally define unitarily equivalent.

Pf. If a strategy is optimal, then all ineq. must be tight:

$$tr(CHSH\ \rho_{AB}) = 2\sqrt{2} \quad \text{and} \quad \|CHSH\| = 2\sqrt{2}.$$

so $\rho_{AB} = \sum p_i |\psi_i\rangle\langle\psi_i|$ with $\langle\psi_i|CHSH|\psi_i\rangle = 2\sqrt{2}$.

So, ρ_{AB} is a linear combination of "pure strategies";

each of which is a $2\sqrt{2}$ eigenvalued eigenvector of CHSH.

Note: $\|CHSH\| = 2\sqrt{2} \iff \|[A_0, A_1]\| = 2$ and $\|[B_0, B_1]\| = 2$.

Does $\|[A_0, A_1]\| = 2$ imply $A_0 A_1 = -A_1 A_0$?