CSE 534 Autumn 2024: Set 5

Instructor: Chinmay Nirkhe
Due date: December 4th, 2024 10:00pm
There are no extensions allowed; contact Chinmay in the case of extreme difficulty.

Instructions: Solutions should be legibly handwritten or typeset. Mathematically rigorous solutions are expected for all problems unless explicitly stated.

You are encouraged to collaborate on problems in small teams but everyone must individually submit solutions. Solutions for the problems may be found online or in textbooks – but do not use them.

For grading purposes, start each problem on a new page.

Problem 1 (5-local Hamiltonian is QMA-hard)**.** In class, we proved that it is QMA-hard to decide whether $\lambda_{\min}(\mathbf{H}) \leq a$ or $\geq b$ when $\mathbf{H}$ is a $O(\log N)$-local Hamiltonian acting on N qubits and $a \geq b + \Omega(1/N^3)$. In this problem, you will improve the locality from $O(\log N)$ to 5.

The main transformation involved will be to switch the time register from *binary* to *unary* and an improved analysis of the spectral gap of the sum of Hamiltonian terms.

1. (**4 points**) Construct a 2-local commuting Hamiltonian $\mathbf{H}_{\text{stab}}$ acting on T-qubits such that the groundspace of $\mathbf{H}_{\text{stab}}$ equals

$$\mathcal{S} \stackrel{\text{def}}{=} \operatorname{span}\left\{ \left|0^T\right\rangle, \left|1, 0^{T-1}\right\rangle, \left|1^2, 0^{T-2}\right\rangle, \left|1^3, 0^{T-3}\right\rangle, \dots, \left|1^T\right\rangle \right\}. \tag{1}$$

   Argue that the first non-zero eigenvalue of $\mathbf{H}_{\text{stab}}$ is 1.

2. (**6 points**) Your solution to the previous problem generates a subspace of dimension $T+1$ within $(\mathbb{C}^2)^{\otimes T}$. Replace the time register in the $O(\log N)$-local Hamiltonian construction accordingly. Adjust the terms h_t, $\mathbf{H}_{\text{in}}$, and $\mathbf{H}_{\text{out}}$ to use the new time register and to be 5 local.

Problem 2. In class, we proved that when a quantum circuit has max accepting probability $\leq \epsilon$, the resulting Hamiltonian $\mathbf{H}$ had a minimum eigenvalue of $\lambda_{\min} \geq \Omega(1/T^3)$. This is also the case for the 5-local Hamiltonian (analysis is identical). The analysis can be improved to generate a lower bound of $\Omega(1/T^2)$ but we do not include that here.

(**8 points**) Construct a state $|\varphi\rangle$ such that $\langle\varphi|\mathbf{H}|\varphi\rangle \leq O(1/T^2)$.

It may be easier to construct a state $|\mu\rangle$ of small energy for $V^\dagger \mathbf{H} V$ where V is the unitary derived in class for analyzing the Hamiltonian. Write $|\mu\rangle = \sum_t |t\rangle |\mu_t\rangle$ and choose all the vectors $|\mu_t\rangle$ to be parallel but of monotonically decreasing lengths according to a trignometric function.

Problem 3.
"A man with a watch knows what time it is. A man with two watches is never sure." - Segal's law.

1. (**4 points**) By writing out the stabilizers, construct a 4-qubit code encoding 1 qubit which *detects* all 1-qubit errors. Prove that it is unable to correct them.
2. (**2 points**) Come up with two stabilizers which commute with the stabilizers for the 4-qubit code but anti-commute with each other. What do these mathematically represent? Give a basis for the code that matches.

Problem 4 (Logical flip or correctable error?)**.** In class, we proved that the Shor code can correct any single qubit error. While it cannot correct all two-qubit errors, there are some which it can correct. Recall the Shor code is stabilized by

$$\begin{array}{rccccccccc}
S_1 = & Z & Z & I & I & I & I & I & I & I \\
S_2 = & I & Z & Z & I & I & I & I & I & I \\
S_3 = & I & I & I & Z & Z & I & I & I & I \\
S_4 = & I & I & I & I & Z & Z & I & I & I \\
S_5 = & I & I & I & I & I & I & Z & Z & I \\
S_6 = & I & I & I & I & I & I & I & Z & Z \\
S_7 = & X & X & X & X & X & X & I & I & I \\
S_8 = & I & I & I & X & X & X & X & X & X \\
\hline
\overline{X} = & X & X & X & X & X & X & X & X & X \\
\overline{Z} = & Z & Z & Z & Z & Z & Z & Z & Z & Z
\end{array} \tag{2}$$

1. (**5 points**) Which of these errors are correctable? (Give an explanation as to why.)

$$(a)\ E_1 = X_1X_3,\ (b)\ E_2 = X_2X_7,\ (c)\ E_3 = X_5Z_6,\ (d)\ E_4 = Z_5Z_6,\ (e)\ E_5 = Y_2Z_8 \tag{3}$$

2. (**3 points**) If the error E_i from the previous part is *not* correctable and we applied the correction $\mathrm{Dec} \circ E_i \circ \mathtt{Enc}(|\psi\rangle)$, what would be the logical transformation that gets applied to $|\psi\rangle$?

Problem 5 (Circuit depth of error-correcting codes)**.**

1. (**2 points**) Let U be a tensor product of $n/2$ 2-qubit unitaries with any connectivity. Prove that if C is a $[[n, k, d]]$ codespace, then $C' = UCU^\dagger$ is a $[[n, k, d/2]]$ codespace.

2. (**6 points**) A unitary U acting on n-qubits is called a depth T unitary if $U = U_T U_{T-1} \dots U_1$ where each U_t is a tensor product of $n/2$ 2-qubit unitaries with any connectivity.

   Let $\mathcal{S}$ be the subspace defined by projector $\mathbb{I}_{2^k} \otimes |0\rangle\langle 0|^{\otimes(n-k)}$. A n-qubit unitary $\texttt{Enc}$ is called an encoding circuit for the $[[n, k, d]]$ code C if $\texttt{Enc} \cdot \mathcal{S} \cdot \texttt{Enc}^\dagger = C$. Prove that $\texttt{Enc}$ must be a depth T unitary for $T \geq \Omega(\log d)$.

3. (**6 points**) Let C be a $[[n, k, d]]$ quantum code and let S be any subset $\subset [n]$ such that $|S| = d - 1$. Prove from the Knill-Laflamme conditions, that for any two codewords $|\psi_1\rangle, |\psi_2\rangle \in C$, that the reduced density matrices on S are equal:

$$\mathrm{tr}_{[n]\setminus S}(\psi_1) = \mathrm{tr}_{[n]\setminus S}(\psi_2). \tag{4}$$

   Prove that for any ℓ-local Hamiltonian $\mathbf{H}$ for $\ell < d$, that $\langle\psi|\mathbf{H}|\psi\rangle$ is an invariant over all $|\psi\rangle \in C$.