CSE 534 Autumn 2024: Set 1

Instructor: Chinmay Nirkhe
Due date: October 2nd, 2024 10:00pm

Instructions: Solutions should be legibly handwritten or typset. Mathematically rigorous solutions are expected for all problems unless explicitly stated.

You are encouraged to collaborate on problems in small teams but everyone must individually submit solutions. Solutions for the problems may be found online or in textbooks – but do not use them.

For grading purposes, start each problem on a new page.

Problem 1 (Validity of Born's rule)**. (2 points)** For any given quantum state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, show that n-qubit Born's rule yields a valid probability distribution.

Problem 2 (Tensor Products and Bra-ket notation)**.** Let A, C be matrices in $\mathbb{C}^{d_1 \times d_1}$ and B, D be matrices in $\mathbb{C}^{d_2 \times d_2}$. Prove the following identities for **1/2 a point each**. (Assume A and B are invertible for the last identity).

$$A_{ij} = \langle i| A |j\rangle \tag{1a}$$
$$\operatorname{tr}(A) = \sum_i \langle i| A |i\rangle \tag{1b}$$
$$\langle i| AC |j\rangle = \sum_k \langle i| A |k\rangle \cdot \langle k| C |j\rangle . \tag{1c}$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{1d}$$
$$\langle i, k| A \otimes B |j, \ell\rangle = \langle i| A |j\rangle \cdot \langle k| B |\ell\rangle . \tag{1e}$$
$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \tag{1f}$$
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{1g}$$

Problem 3. (2 points) Show that if $|a_1\rangle, \dots, |a_{d-1}\rangle$ are an orthonormal basis for $\mathbb{C}^{d_1}$ and $|b_1\rangle, \dots, |b_{d_2}\rangle$ are an orthonormal basis for $\mathbb{C}^{d_2}$, then the set of tensor product terms $|a_i\rangle \otimes |b_j\rangle$ form an orthonormal basis for $\mathbb{C}^{d_1 d_2}$.

Problem 4 (Swap unitary)**.** the swap unitary **SWAP** is defined as the unique transformation such that for any $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ and $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$,

$$\mathbf{SWAP}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle \,. \tag{2}$$

The $\mathbf{CNOT}_{12}$ unitary can be defined as the unique unitary such that for $a, b \in \{0, 1\}$,

$$\mathrm{CNOT}_{12} \, |a\rangle_1 \otimes |b\rangle_2 = |a\rangle_1 \otimes |a \oplus b\rangle_2 \,. \tag{3}$$

(2 points) Prove that the swap unitary for single qubits can be defined as

$$\mathrm{SWAP} = \mathbf{CNOT}_{12} \cdot \mathbf{CNOT}_{21} \cdot \mathbf{CNOT}_{12}. \tag{4}$$

Problem 5 (Partial measurements)**.** In the second lecture, we saw/will see an axiom for a partial measurement of only the 1st qubit of a n-qubit quantum state $|\psi\rangle$. Namely that the probability of measuring $i \in \{0, 1\}$ is $\||\psi_i\rangle\|^2$ and the resulting state is $|i\rangle \otimes |\psi_i\rangle / \||\psi_i\rangle\|$ where

$$|\psi_i\rangle \stackrel{\text{def}}{=} (\langle i| \otimes \mathbb{I}_{2^{n-1}}) \, |\psi\rangle \,; \text{ equivalently, } |\psi\rangle = |0\rangle \, |\psi_0\rangle + |1\rangle \, |\psi_1\rangle \,. \tag{5}$$

1. **(1 point)** How can measurement of the first qubit be used to emulate measurement of any other single qubit?
2. **(2 points)** Notice that after measurement, the first qubit it unentangled from the remaining $n - 1$. What is the state of the system after measuring out the first k qubits (sequentially) and getting outcomes $y_1, \ldots, y_k$? What is the probability of this event?
3. **(2 points)** Argue that the order of measurement of the k qubits did not matter.
   Hint: Use that $A \otimes \mathbb{I}$ and $\mathbb{I} \otimes B$ as matrices always commute.

Problem 6 (A very large Hilbert space)**.** High-dimensional geometry is weird. We will show that in a n-qubit Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, we can find a doubly-exponential sized set of normal vectors that are all pairwise ϵ-orthogonal for any constant $\epsilon > 0$.

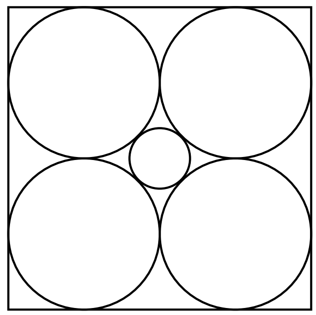
1. **(1 point)** What is the maximal number of pairwise perfectly orthogonal states in $\mathcal{H}$?
2. **(2 points)** For every function $f : \{0, 1\}^n \to \{0, 1\}$ define the *phase state*

$$|\psi_f\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \, |x\rangle \,. \tag{6}$$

   For a fixed f and uniformly random g, calculate an upper bound on the probability that $|\psi_f\rangle$ and $|\psi_g\rangle$ are *not* ϵ-orthogonal – i.e. $\left|\langle \psi_f | \psi_g \rangle\right| > \epsilon$.
3. **(2 points)** Using a greedy strategy, generate (for constant ϵ) a doubly-exponentially large set of phase states that are pairwise ϵ-orthogonal. What is the dependence on the set size in terms of ϵ and n?

It is interesting to note that there does not exist any asymptotically larger set of ϵ-orthogonal vectors within $\mathcal{H}$. This can be proven by showing that the problem we are considering is a special case of the *Johnson-Lindenstrauss lemma*. The optimality of which was shown by Green Larsen and Nelson.

Problem 7 (Higher dimensional shenanigans)**.** Place 4 circles of radius 1 inside a square box of side length 4 such that the circles don't overlap and only touch each other (as shown). Now place another circle centered at the center of the box that is tangent to all 4 circles.



1. **(1 point)** What is the radius of the center circle?
2. **(1 point)** The three-dimensional analog is 8 spheres of radius 1 inside a cube of side length 4. A center sphere is placed tangent to the 8 spheres. What is the radius of the center sphere?
3. **(2 points)** The n-dimensional analog consists of placing 2^n many $(n-1)$-spheres of radius 1 inside a n-cube of side length 4. A center sphere is placed tangent to the 2^n spheres. What is the radius of the center sphere?

   Calculate the $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the center $(n-1)$-sphere still only touches the other spheres but bleeds outside the n-cube.

Problem 8 (Reversible classical computation)**.**
This problem is optional and not graded. However, I expect you know how to solve it.

You probably know that every boolean function $f : \{0,1\}^n \to \{0,1\}^m$ can be computed by a classical boolean circuit consisting of a few basic operators: NOT, AND, COPY, and initializing extra bits as zeroes. The function f, in general, will not be reversible – meaning that given $y \in \operatorname{Im} f$, there may not be a way to compute x such that $f(x) = y$, as the x may not be unique.

However, the function $f' : \{0,1\}^n \to \{0,1\}^{n+m}$ defined by

$$f'(x, y) \stackrel{\text{def}}{=} (x, f(x) \oplus y) \tag{7}$$

is a reversible function. A boolean gate on k bits is reversible if it can be represented by a 2^k-sized permutation matrix.

1. The reversible version of the AND gate is the 3-bit gate defined by

$$(x, y, z) \mapsto (x, y, x \cdot y \oplus z) \text{ for } x, y, z \in \{0, 1\}. \tag{8}$$

   What is the 8×8 matrix representation of this gate?

2. Construct a circuit using only reversible 1-,2-, or 3-bit reversible gates that computes f'. Your circuit will likely need additional bits initialized as zeroes. First construct a function g that generates output $(x, f(x) \oplus y, \text{junk})$ from input $(x, y, 0 \dots 0)$. Then show how you can run the circuit in reverse and "uncompute" the junk so that your final output is $(x, f(x) \oplus y, 0 \dots 0)$.

3. How many gates were required to compute f' with respect to the number of gates required to compute f?