Lecture 18

Nov 26, 2024

Typically, when people talk about classical error correction they are talking about linear codes.

$k = \dim C$ and $C = \ker A \leftarrow$ check matrix.

Notation: $C = [n, k, d]$ code with locality ℓ if $C = \ker A$ with A being ℓ-row & -column sparse.

Ex. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

$Ax = 0$ equals

$$x_1 \oplus x_2 = x_2 \oplus x_3 = 0.$$

$C = [3, 1, 3]$ code.

$$d = \min_{\substack{x \neq y \\ x, y \in C}} d_H(x, y) = \min_{\substack{x \in C \\ x \neq 0}} |x|.$$

Quantum Codes.

Let $C \subseteq (\mathbb{C}^2)^{\otimes n}$ be a Hilbert space s.t. $\dim C = 2^k$ and for all $E = E'_S \otimes \mathbb{1}_{[n] \setminus S}$ where $|S| < d$, we have

$$\langle \psi_1 | E | \psi_2 \rangle = 0 \quad \text{if} \quad |\psi_1\rangle \perp |\psi_2\rangle.$$

Equiv, dist d is the max d s.t. for all Paulis of size d,

$$\Pi P \Pi = \eta_P \Pi \quad \text{for } \Pi \text{ the projector onto } C.$$

Like prev, we can correct up to distance $\lfloor \frac{d-1}{2} \rfloor$.

Notation: $[[n, k, d]]$ code.

Shor's code is a $[[9, 1, 3]]$ code.

How do we build codes of better parameters?

To do, so we will study a special subclass of codes called Stabilizer codes due to their fundamental relation to Paulis and stabilizers.

Recall stabilizer states as the states defined by linearly indep. and commuting Pauli's $P_1 \dots P_n$.

Well if we only considered $P_1, \dots, P_{n-k}$, there will be a 2^{n-k} dim subspace $\left(\cong (\mathbb{C}^2)^{\otimes k}\right)$ of states s.t.

$$P_i|\psi\rangle = |\psi\rangle.$$

<u>Claim</u> $\operatorname{span}\{|000\rangle, |111\rangle\}$ is defined by Z_1Z_2, Z_2Z_3.

Pf. $$Z_1Z_2\left(\sum_x \alpha_x |x\rangle\right) = \sum_x (-1)^{x_1+x_2}\alpha_x |x\rangle$$

so $\alpha_x = 0$ when $x_1 + x_2 = 1$.

Likewise, $\alpha_x = 0$ when $x_2 + x_3 = 1$. so

$\alpha_x \neq 0$ when $x \in \{000, 111\}$. ✓ $\boxtimes$

Recall notion of measuring w.r.t. an observable.

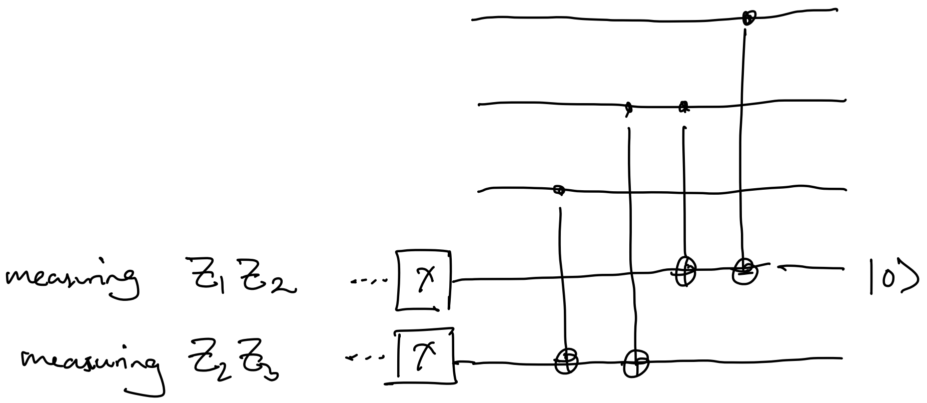
$$M = \underset{\substack{\uparrow \\ +1 \text{ eigenspace}}}{\Lambda_+} - \underset{\substack{\uparrow \\ -1 \text{ eigenspace,}}}{\Lambda_-}$$

then

we have a POVM $\{\Lambda_+, \Lambda_-\}$.

The bit flip code has stabilizers $Z_1 Z_2, Z_2 Z_3$.

Where do these show up?



logical bit flip. $X_1 X_2 X_3 = \bar{X}$

commutes with all stabilizers.

Is there a logical phase flip? Yes. Z_1.

Since there is a 1 qubit phase flip, this cannot correct against phase flip errors.

Phase flip code stabilized by $X_1X_2,\ X_2X_3$.

logical phase flip. $Z_1Z_2Z_3$

logical bit flip. X_1.

What are the stabilizers of Shor code?

$$|i\rangle \mapsto \left(\frac{|0\rangle + (-1)^i|1\rangle}{\sqrt{2}}\right)^{\otimes 3} \mapsto \left(\frac{|000\rangle + (-1)^i|111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

$$X_1X_2,\ X_2X_3$$

$$\downarrow$$

$$X_1X_2X_3X_4X_5X_6$$

$$X_4X_5X_6X_7X_8X_9.$$

$$\downarrow$$

$$(Z_1Z_2,\ Z_2Z_3)$$

$$(Z_4Z_5,\ Z_5Z_6)$$

$$(Z_7Z_8,\ Z_8Z_9)$$

$$\bar{X} = X_1 \cdots X_9,\quad \bar{Z} = Z_1 \cdots Z_9.$$

Are there other logical bit and phase flips?

Table of stabilizers :

$$\begin{array}{lccccccccc}
S_1 = & Z & Z & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
S_2 = & 1 & Z & Z & 1 & 1 & 1 & 1 & 1 & 1 \\
S_3 = & 1 & 1 & 1 & Z & Z & 1 & 1 & 1 & 1 \\
S_4 = & 1 & 1 & 1 & 1 & Z & Z & 1 & 1 & 1 \\
S_5 = & 1 & 1 & 1 & 1 & 1 & 1 & Z & Z & 1 \\
S_6 = & 1 & 1 & 1 & 1 & 1 & 1 & 1 & Z & Z \\
S_7 = & X & X & X & X & X & X & 1 & 1 & 1 \\
S_8 = & 1 & 1 & 1 & X & X & X & X & X & X \\
\bar{X} = & X & X & X & X & X & X & X & X & X \\
\bar{Z} = & Z & Z & Z & Z & Z & Z & Z & Z & Z .
\end{array}$$

How do we correct and detect errors for stabilizer code? Suffices to consider Paulis.

Let C be stabilized by $\langle S_1, \ldots, S_{n-k} \rangle$

Three types of errors: Good, Bad, Ugly.

① Good error. E is a product of stabilizers. Then $E|\psi\rangle = |\psi\rangle$ and nothing changed.

② Bad error. E anticommutes with some S_i.

③ Ugly error. E commutes with all $S_1, \ldots, S_k$ but is outside their span.

Bad errors are detectable. To detect errors, measure each stabilizers S_i. If $ES_i = -S_i E$, then

$$S_i E|\psi\rangle = -ES_i|\psi\rangle = -E|\psi\rangle \text{ for } |\psi\rangle \in C,$$

Therefore S_i measurement outputs -1.

Ugly errors are logical transforms. They are undetectable as every stabilizer will measure $+1$ but the state changes.

Ex. For bit flip code

good	Z	Z	I
bad	I	I	X
ugly	X	X	X

Let $\mathcal{G} = \langle S_1, \dots, S_k \rangle$

The centralizer $C(G) = C_{\mathcal{P}_n}(G)$ is the set

$$\{ P \in \mathcal{P}_n \mid \forall g \in G,\ Pg = gP \}.$$

The set of Paulis which commute with all of G.

Then the set of errors can be characterized by

$$\text{good} = \mathcal{G}$$
$$\text{bad} = \mathcal{P}_n \setminus C(\mathcal{G})$$
$$\text{ugly} = C(\mathcal{G}) \setminus \mathcal{G}.$$

A stabilizer code has distance d, if every Pauli error of size $< d$ is either good or bad

(equiv. trivial or correctable).

<u>Thm</u> For a stabilizer code on n-qubits with $n-k$ independent Pauli stabilizers $S_1, \dots, S_{n-k}$, let $\mathcal{G} = \langle S_1, \dots, S_k \rangle$.

Then the rate of the code is k and the distance is the minimum size of a Pauli $\in C_{P_n}(\mathcal{G}) \setminus \mathcal{G}$.

Next : Kitaev's toric code. A construction of an error correcting code with local checks and distance growing with n.

Toric code is a special case of

Caulderbank - Shor - Steane (CSS) codes

where each stabilizer generator is either X-type or Z-type.

X-type $= X^a \leftarrow$ tensor product of only X terms.

Z-type $= Z^b \leftarrow$ tensor product of only Z terms.

Shor's code is also CSS.

Obs X-type checks detect for $\underbrace{Z\text{-errors}}_{\text{only tensor product of } Z \text{ and } \mathbb{1}.}$ and Z-type checks detect for X-errors.

What is the smallest Z-error that is logical?

It's the smallest element of $C_{P^n}(G) \setminus G$ which only consists of Z-terms.

goal is to design a code s.t. all small Z-errors either are

ⓐ detected by the X-checks

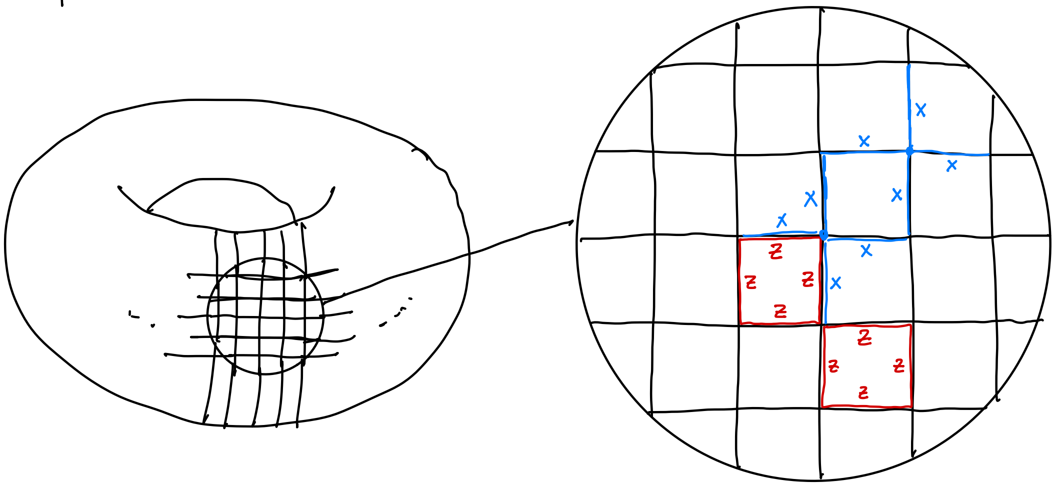
ⓑ product of the Z-checks and (therefore, trivial as it acts like a stabilizer).

For $E \subseteq [n]$, let $Z_E = \mathbb{1} \otimes \mathbb{1} \otimes \dots \underbrace{Z \otimes Z \otimes Z}_{\text{locations indicated by } E.} \otimes \mathbb{1}$

An error Z_E is detected by a stabilizer X_A if $A \cdot E = 1$. Equiv., the size of the intersection $|A \cap E|$ is odd.

So, the "Z-distance" of the code is the smallest size error Z_E where intersection with every X-check X_A is even but Z_E is not a product of the Z-checks.

Place qubits on the edges of a grid-discretization of a torus.



For every face f, place a check Z_f which equals $\underbrace{Z \otimes Z \otimes Z \otimes Z}_{\text{edges touching } f.}$

And for every vertex v, place a check X_v which equals $\underbrace{X \otimes X \otimes X \otimes X}_{\text{edges touching } v.}$

All stabilizers commute as the intersection of a face f and a vertex v is either 0 or 2.

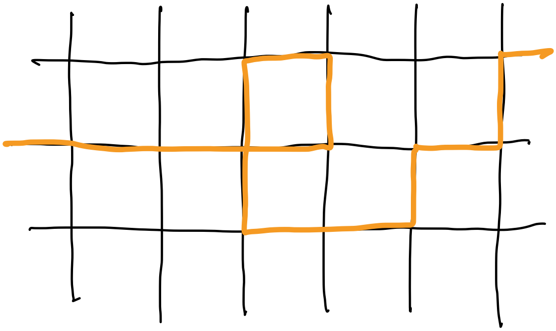
Two observations:

① A Z-error Z_E commutes with every X-check X_v iff $E =$ union of cycles.

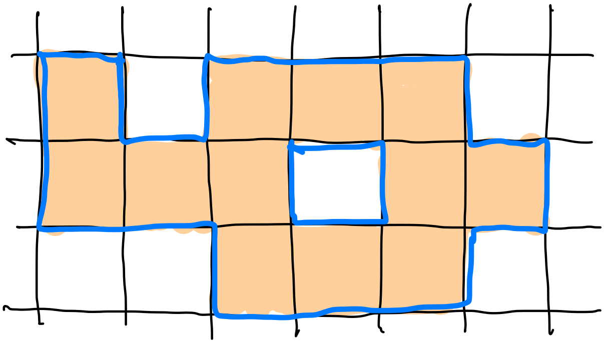
Pf. Use the edges $\in E$ to draw a graph (V, E).

The degree of every vertex is 0, 2, or 4. So the graph can be decomposed as a union of cycles.

"for every edge in, there is an edge out"



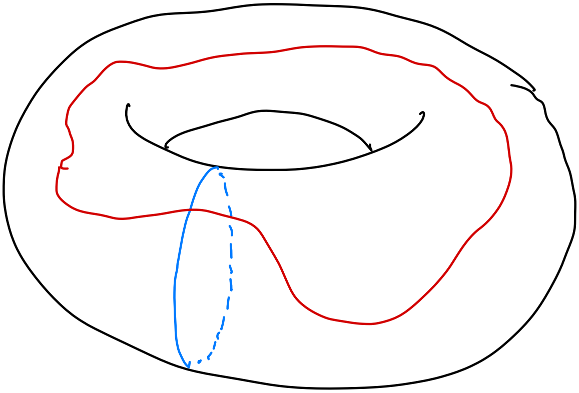
② The product of Z-checks is the boundary of a collection of faces.



What is the difference between cycles and boundaries?

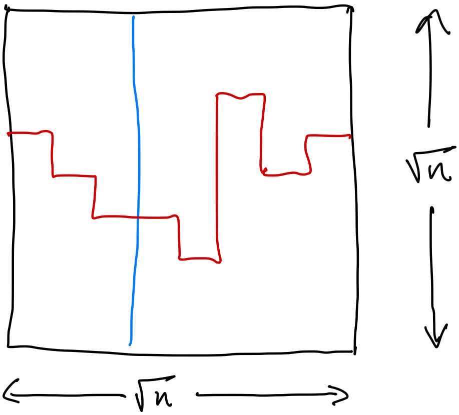
All boundaries are cycles but not all cycles are boundaries.

Example.



cycles which are not boundaries.

To see this its often easiest to unfold the torus.



Identify the top & bottom and identify the sides.

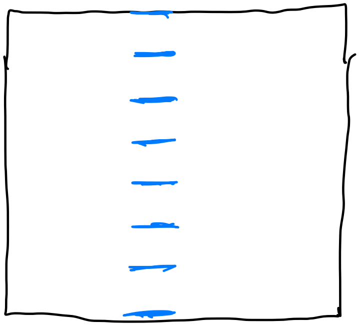
Cycles \ Boundaries = "non-trivial loops".

What is the shortest non-trivial loop?

Length = $\sqrt{n}$.

So, the Z-distance will be $\sqrt{n}$.

The X- distance is also $\sqrt{n}$ by a similar argument.



A non-trivial loop through the faces.

" co-cycles" \ "co-boundaries"

Correcting a general Pauli error.

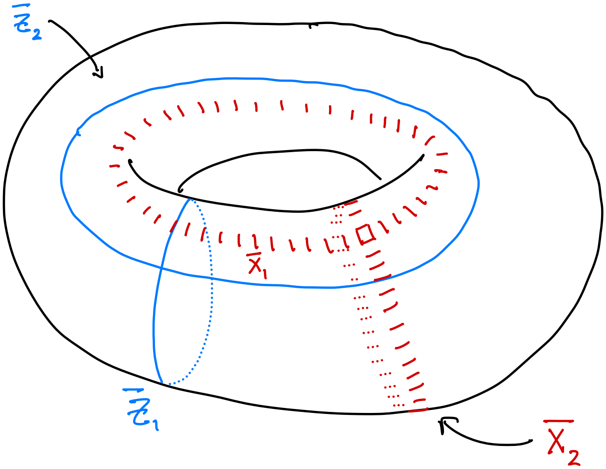
$$P = X_E Z_{E'} .$$

Since checks are only X- or Z-type, P anticommutes with X_v iff $|E' \cap v|$ is odd and with Z_f iff $|E \cap f|$ is odd.

Therefore, correctable if X_E and $Z_{E'}$ are both seperately correctable.

What are the logical transformations for this code?

They will correspond to non-trivial loops.



Notice $\bar{X}_1$ and $\bar{Z}_1$ share an edge and therefore anticommute. Likewise $\bar{X}_2$ and $\bar{Z}_2$ anticommute. Other relations are commutation.

These logical operators are defined up to stabilizers.

By these relations, these define 2 logical qubits.

There are multiple pfs that these are the only logical qubits such as counting the number of independent stabilizerss.

So, this is a $[[n, 2, \Omega(\sqrt{n})]]$ code.

For the longest time, this was the best known code. Today, we have constructions of $[[n, \Omega(n), \Omega(n)]]$ codes.

Lastly,

A rotated basis picture on stabilizer codes.

Let $\mathcal{G} = \langle S_1, \dots, S_{n-k} \rangle$ for indep. stabilizers S_i.

Then $\exists$ unitary V s.t. $V S_i V^\dagger = Z_i$.

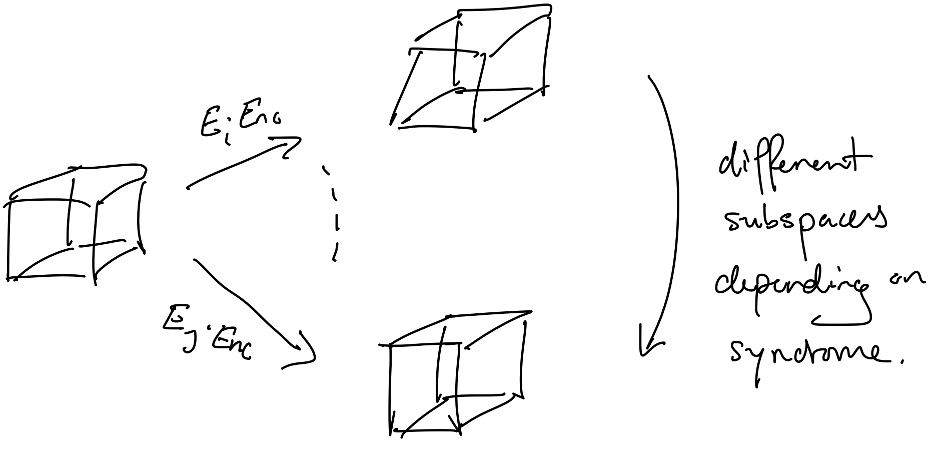
Then $V C V^\dagger = |0\rangle^{\otimes n-k} \otimes (\mathbb{C}^2)^{\otimes k}$

V is therefore the Encoding circuit.

Furthermore, if we measure the syndrome and get out $\vec{s} \in \{0,1\}^{n-k}$, then the state lies in

$$|\vec{s}\rangle \otimes (\mathbb{C}^2)^{\otimes n-k}.$$

So,


$E_i \cdot Enc$
$E_j \cdot Enc$
different subspaces depending on syndrome.