Lecture 17

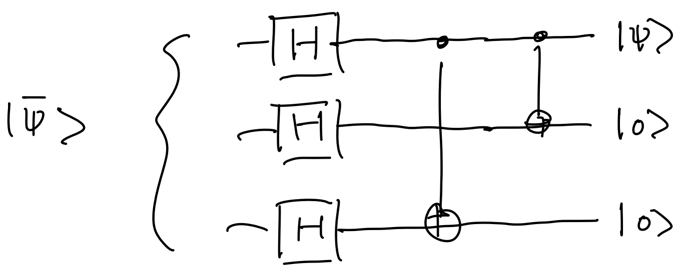
Nov 21, 2024

Correcting only phase errors:

Note $HZH = X$. Phase errors are bit flip errors in a different basis.

$$|0\rangle \mapsto |+++\rangle$$
$$|1\rangle \mapsto |---\rangle.$$

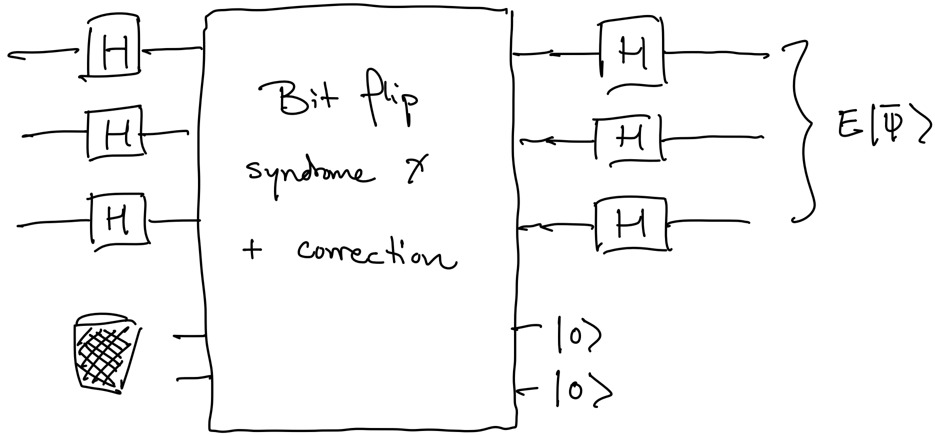
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto |\bar{\psi}\rangle = \alpha|+++\rangle + \beta|---\rangle.$$



Z error on 3rd qubit. Then

$$\alpha|+++\rangle + \beta|---\rangle \xmapsto{Z_3} \underbrace{\alpha|++-\rangle + \beta|--+\rangle}_{= H^{\otimes 3}\,\alpha|001\rangle + \beta|110\rangle}.$$

So correction circuit is easy to concoct.



Note if we encode in phase flip code but X_1 occurs then

$$\alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle \xmapsto{X_1} \alpha|{+}{+}{+}\rangle - \beta|{-}{-}{-}\rangle$$

New state will have syndrome $(0,0)$ for phase flip code and corresponds to applying Z on underlying logical information.

Same with phase flip for bit flip encoding.

How do we combine to get both error corrections?

Ans: Encode in one code and then in the other.

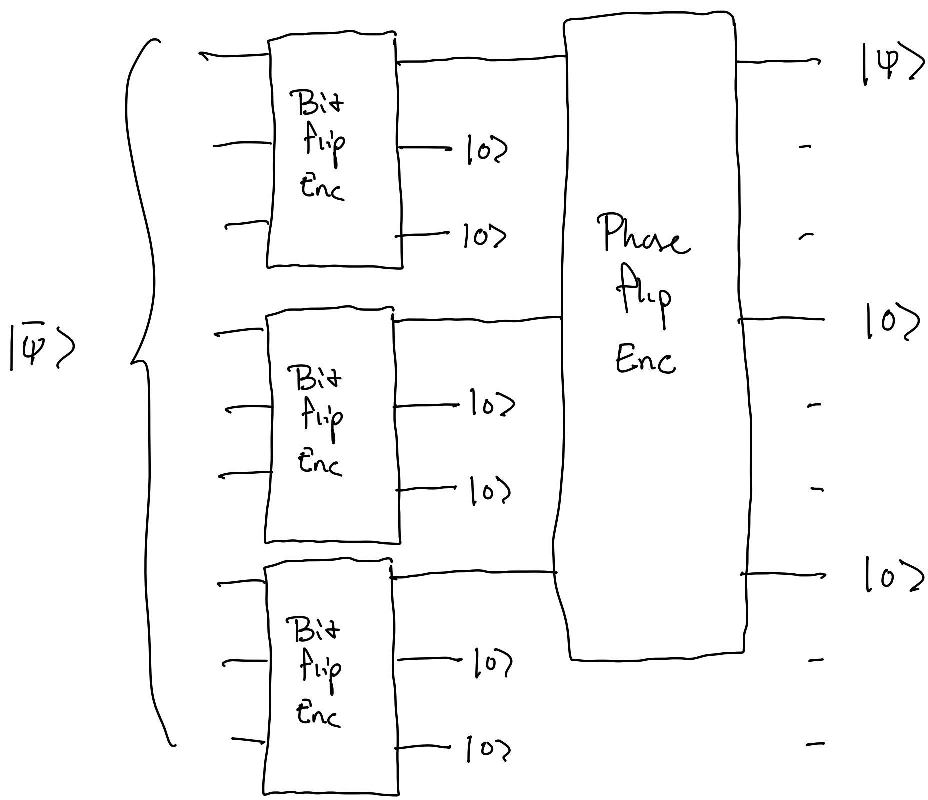
$$|0\rangle \longmapsto |+\rangle^{\otimes 3} \longmapsto \frac{1}{\sqrt{8}}\left(|000\rangle + |111\rangle\right)^{\otimes 3}$$

$$|1\rangle \longmapsto |-\rangle^{\otimes 3} \longmapsto \frac{1}{\sqrt{8}}\left(|000\rangle - |111\rangle\right)^{\otimes 3}$$

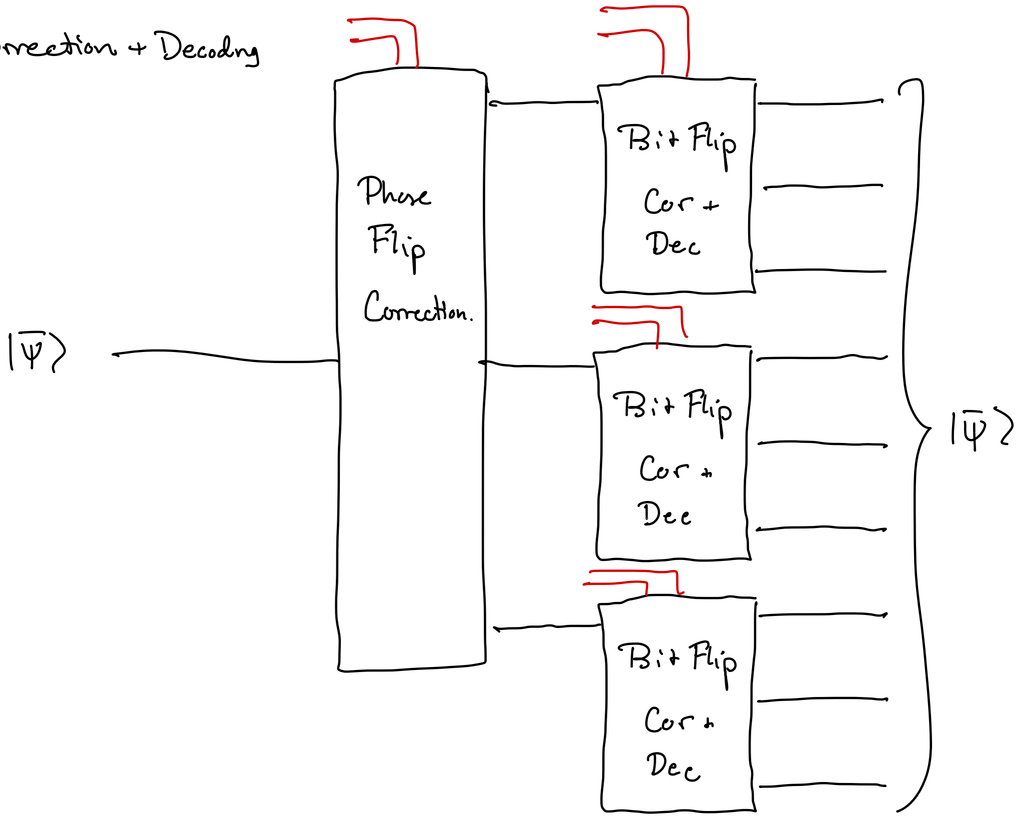
Could have also done in the other order

But has different correction operators.

Encoding:



Correction + Decoding



This corrects any 1 bit flip error per triple and 1 phase flip error. To correct bit + phase flip error in the same qubit, both syndromes check.

<u>Fact</u>: Every Hermitian $H \in \mathbb{C}^{2\times 2}$ equals

$$H = \alpha \mathbb{1} + \beta X + \gamma (XZ) + \delta Z.$$

Next lectures:

1. What about infinite family of errors?

   (Rough answer: Pauli errors form a basis for all errors)

2. Can we correct without decoding? (Yes)

3. How do we encode more than 1 qubit of logical information?

   Is there a general theory of why this is working?

A general theory of q. error correction.

(Slides as there are a lot of drawings)

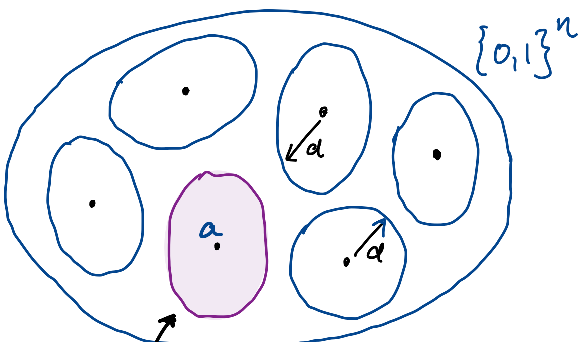
Part 2 (of Lec 17)

An (abridged) theory of classical error correction.

A subset $C \subseteq \{0,1\}^n$ is a code encoding k bits into n bits if $|C| = 2^k$.

The distance of the code is $d = \min_{\substack{x, y \in C \\ x \neq y}} d_H(x, y)$

the min number of bits required required to flip some $x \in C$ to $y \in C$. A code of distance d can correct $\lfloor \frac{d-1}{2} \rfloor$ errors.



all words that correct to a

Typically, when people talk about classical error correction they are talking about linear codes.

$k = \dim C$ and $C = \ker A \leftarrow$ check matrix.

Notation: $C = [n, k, d]$ code with locality ℓ if $C = \ker A$ with A being ℓ-row & -column sparse.

Ex. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. $Ax = 0$ equals $x_1 \oplus x_2 = x_2 \oplus x_3 = 0$.

$C = [3, 1, 3]$ code.

$$d = \min_{\substack{x \neq y \\ x, y \in C}} d_H(x, y) = \min_{\substack{x \in C \\ x \neq 0}} |x|.$$