Lecture 17 (Slides)

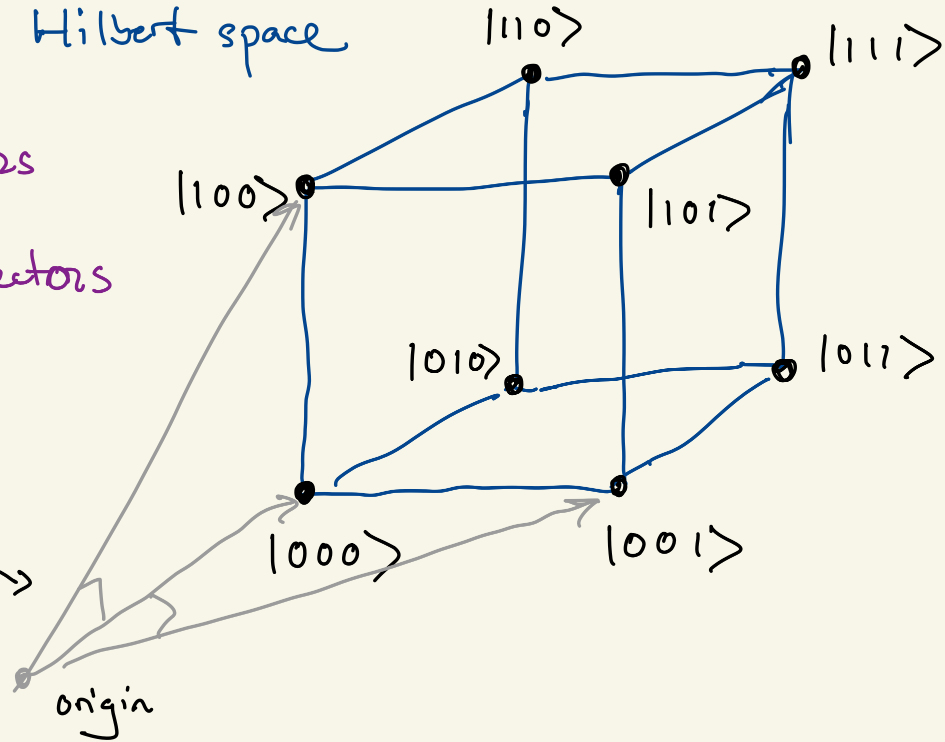
Nov 21, 2024

my drawings of high-dimensional spaces

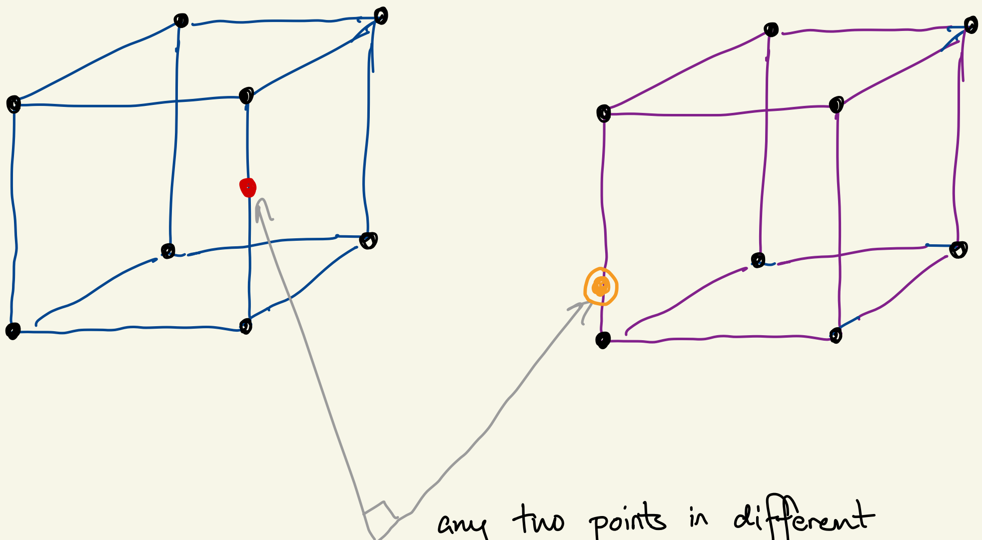
3 - qubit = 2^3 dim Hilbert space

In my drawings, corners represent orthogonal vectors

denoting that both 90° angles



Direct sum of Hilbert spaces

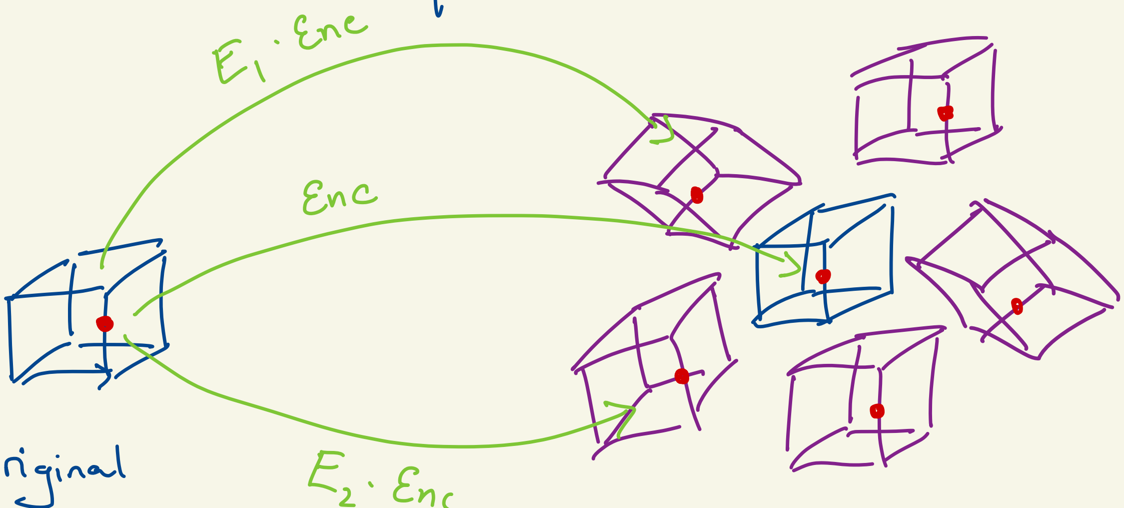


any two points in different shapes are orthogonal.

Ok, now to correcting errors

The packing of Hilbert spaces perspective

goal: correct against unitary errors $E_1, E_2, \ldots, E_J$ (for now)

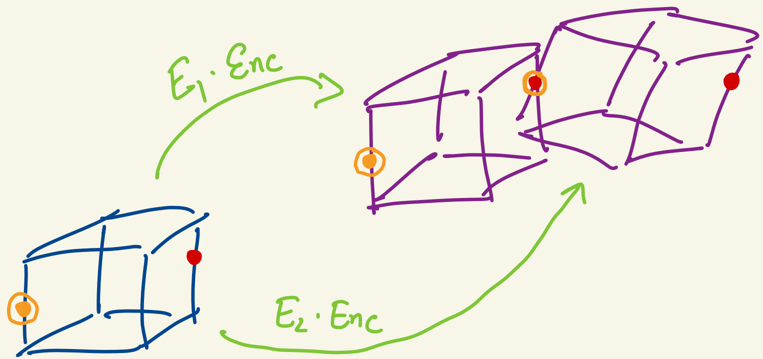


original Hilbert space

morally, we need orthogonal spaces per error.

Similar drawing first appears in Nielsen & Chuang.

Simple example of a <u>non</u> - code

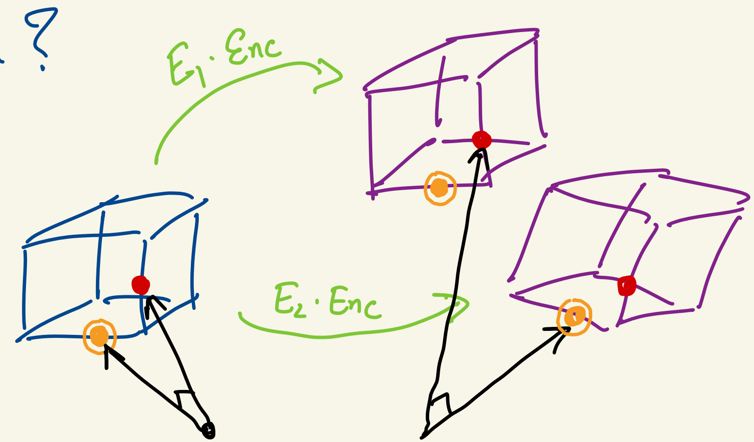


$$\text{i.e.} \quad E_1 \cdot Enc(|\bullet\rangle) = E_2 \cdot Enc(|\odot\rangle)$$

$\Rightarrow$ cannot distinguish these errors.

Actually, stronger statement: If $|\bullet\rangle \perp |\odot\rangle$, then these states should be orthogonal.

Why orthogonal?



Fact 2 states $|a\rangle$ and $|b\rangle$ are perfectly distinguishable

iff $|a\rangle \perp |b\rangle$ (orthogonal vectors)

Notice: only correcting E_1 & E_2 if we can distinguish

$E_1 \cdot Enc(|\bullet\rangle)$ and $E_2 \cdot Enc(|\odot\rangle)$

$\Rightarrow$ These vectors are orthogonal.

It would be too much to ask that

$E_1 \cdot \mathrm{Enc}(|\bullet\rangle)$ and $E_2 \cdot \mathrm{Enc}(|\bullet\rangle)$ are orthogonal.

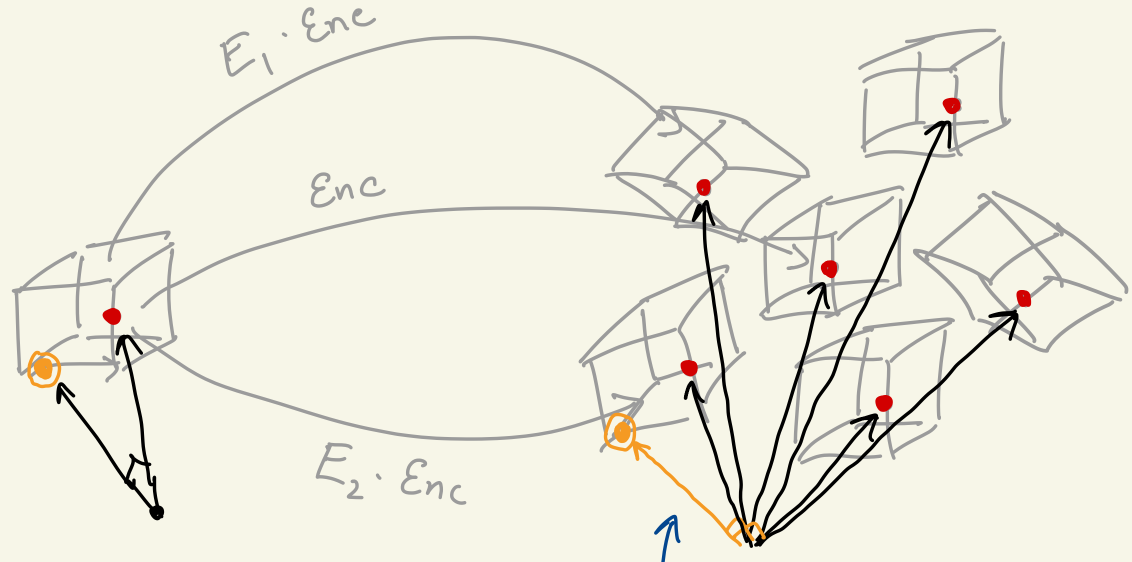
Pf: consider $E_1 \approx_\epsilon E_2$.

By linearity, these states must be close. $\square$

But for some errors E_1, E_2, they will be orthogonal.

Morally, these errors will form a "basis" for the set of errors we can correct.

The benefit of orthogonality:



$E_2 \cdot Enc(|\odot\rangle)$ is orthogonal to every

$E_j \cdot Enc(|\bullet\rangle)$

If $|\bullet\rangle \perp |\odot\rangle$, then

$$\Rightarrow \operatorname{span}_{j_1}\{E_{j_1}\cdot \mathrm{Enc}(|\bullet\rangle)\} \perp \operatorname{span}_{j_2}\{E_{j_2}\cdot \mathrm{Enc}(|\odot\rangle)\}$$

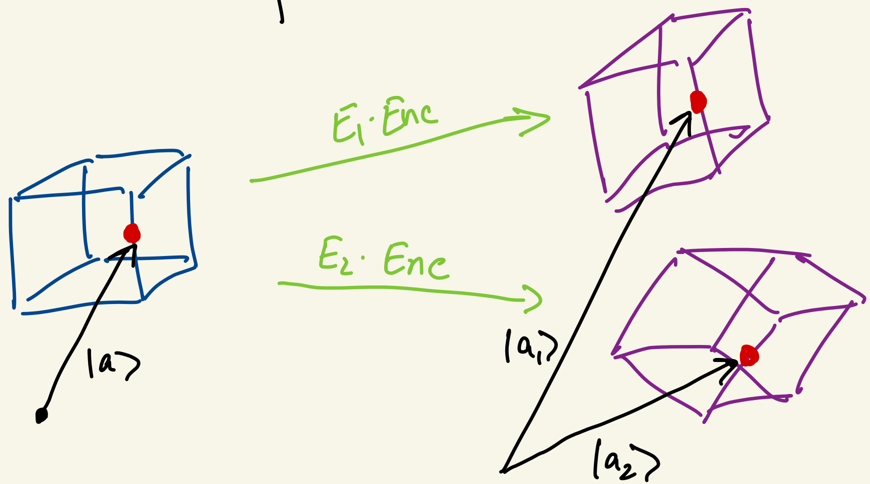
$\Rightarrow$ correcting against errors $E_1, \dots, E_J$ implies correcting against unitary errors in their span.

$\therefore$ Suffices to prove my error-correction properties for a "basis" of the errors. Rest follows directly necessarily.

Why prev. only discussed correcting bit-flip (X) and phase-flip (Z) errors.

A basis for the set of errors.

Exercise :



Show that $\langle a_1 | a_2 \rangle = \eta_{12}$, an invariant that only depends on E_1, E_2 and not the state $|a\rangle$.

Hint: Use the no cloning theorem.

Why does this yield a notion of a basis for the space of errors?

If we can correct all errors $E \in \mathcal{E}$, consider a basis s.t.

$$\operatorname*{span}_{E \in \mathcal{E}} \{ E \cdot Enc|a\rangle \} = \operatorname*{span}_{E_1, \dots, E_J} \{ E_i \, Enc|a\rangle \}.$$

By exercise, for any other state $|b\rangle$,

$$\operatorname*{span}_{E \in \mathcal{E}} \{ E \cdot Enc|b\rangle \} = \operatorname*{span}_{E_1, \dots, E_J} \{ E_i \, Enc|b\rangle \}.$$

$\Rightarrow$ gives natural notion of a basis $E_1 \dots E_J$ for $\mathcal{E}$.

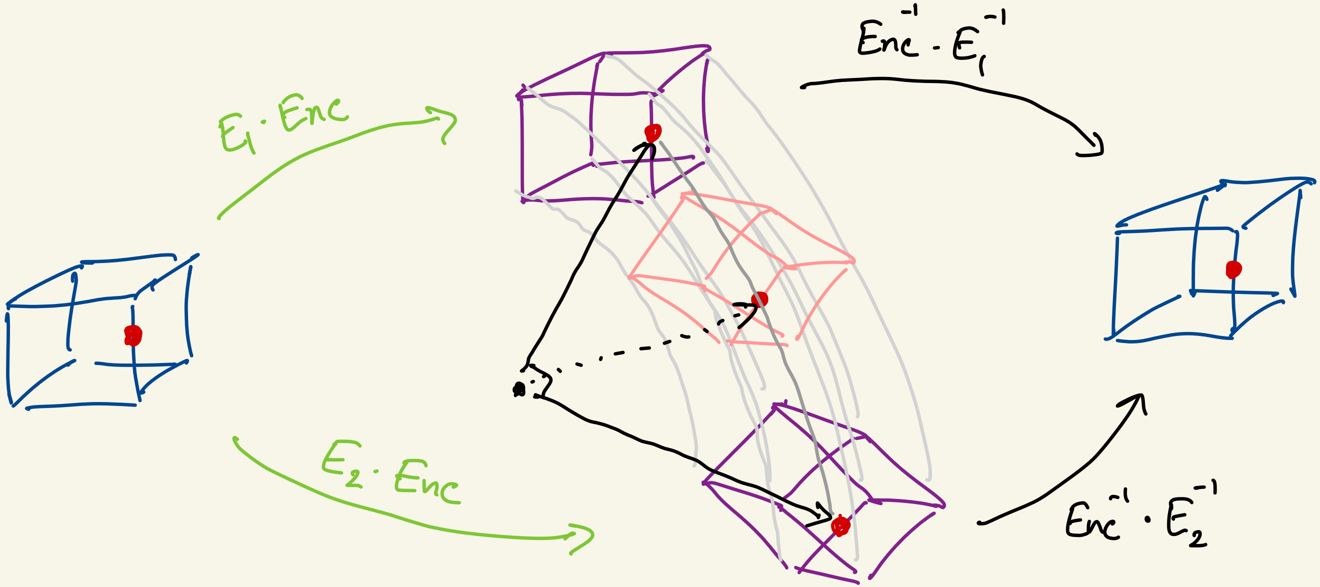
Equiv., can define an inner product on correctable errors

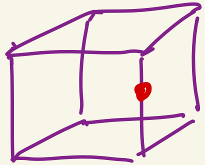
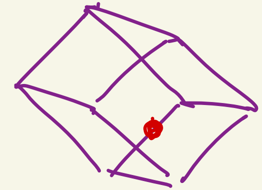
$$\langle E_i, E_j \rangle \overset{\text{def}}{=} \eta_{ij}$$

$$\overset{\text{def}}{=} \langle a | \mathrm{Enc}^\dagger E_i^\dagger E_j \, \mathrm{Enc} | a \rangle \quad \text{for any } |a\rangle.$$

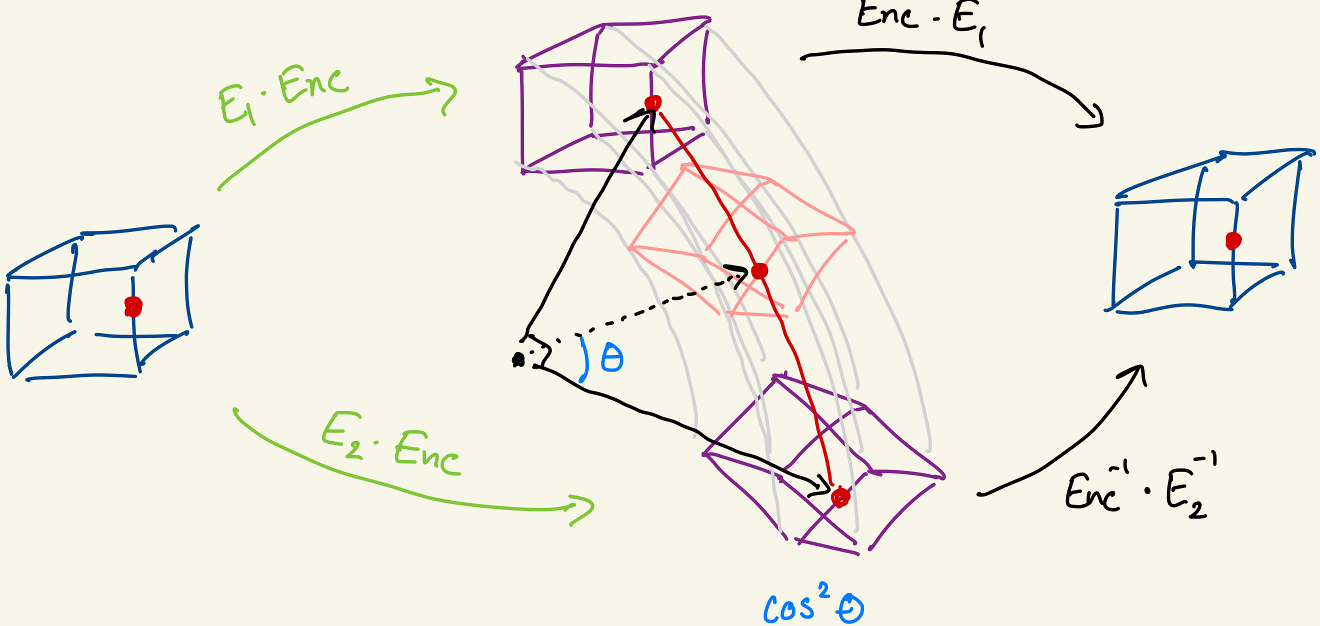
The basis we found is a basis with respect to this inner product.

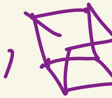
Ok, but does the decoding channel/algorithm look like for the set of continuous errors?

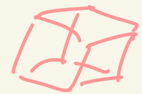


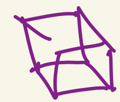
Note that  and  are orthogonal

(assumption that they are a basis for errors)



∃ a measurement perfectly distinguishing the , 

errors. If we measure  error using this,

it collapses to either  or . Plus, we know

$\sin^2\theta$ $\cos^2\theta$

which one!

Gives decoding procedure for a continuous space of errors from a decoding procedure for discrete set.

Generalized error correction procedure:

① Measure syndrome. i.e. collapse cont. error to a basis error.

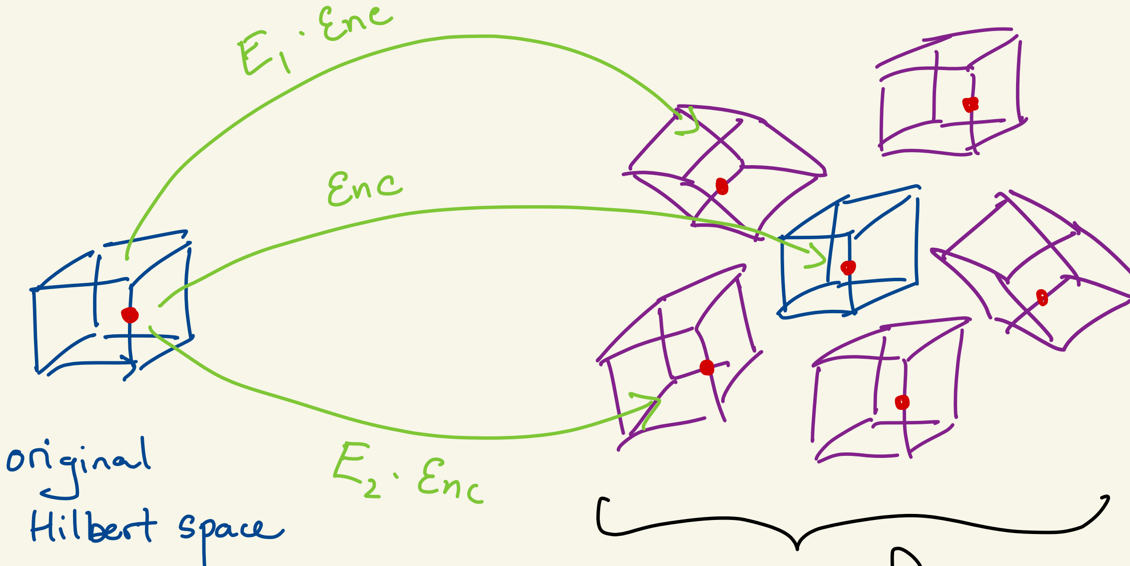
syndrome = name of basis error.

② correct error based on syndrome.

Step ① is non-unitary and Step ② is unitary.

"destructive" "information preserving"

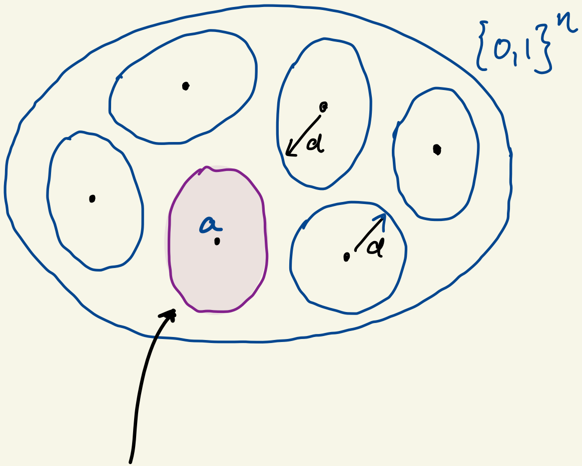
error-correction is a controlled destructive process.



Decomposition of space into direct sum from error basis.

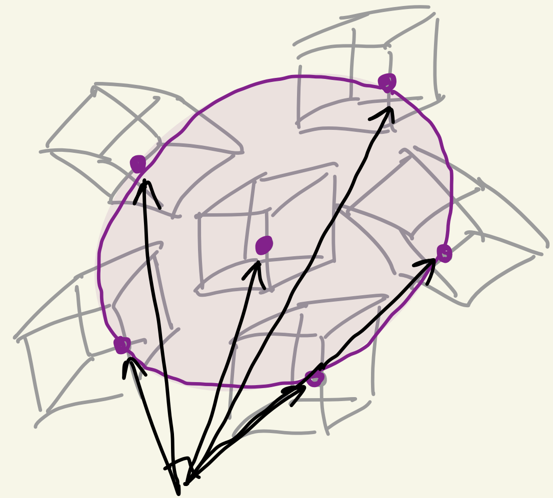
Can also correct error channels whose elements $\in \mathcal{E}$.

Classical picture



all words that correct to a

Quantum picture



subspace of states that correct to $|a\rangle$

The Knill-Laflamme conditions

Mathematically capture all the orthogonality conditions we drew in the cartoons.

Let C be a quantum code. i.e. C = image of encoding map.

Let Π be the projector onto subspace C.

Then, C corrects $\{E_i\}$ if

coefficients s.t. $\eta_{ij} = \eta_{ji}^{\dagger}$.

$$\boxed{\Pi E_i^{\dagger} E_j \Pi = \eta_{ij} \Pi}$$

(the inner product between the error spaces)

Not hard to show this is equivalent to $\forall\ |a\rangle, |b\rangle$,

$$\langle a | Enc^\dagger E_1^\dagger E_2 Enc | b \rangle = \langle a | b \rangle \cdot \underbrace{\langle E_1, E_2 \rangle}_{\eta_{ij} \text{ prev. defined.}}$$

Read Nielsen & Chuang Thm 10.1 for formal pf and explicit construction of the recovery channel.

But morally, its the same as the pictoral argument we've drawn so far.

Correcting errors of size d.

Error of size d: E can be written as $\underbrace{E'}_{d} \otimes \underbrace{\mathbb{1}}_{n-d \text{ qubits}}$

Correcting all errors of size d equiv. to

Correcting all Pauli (X-, Y-, Z-type) errors of size d.

equiv to. correcting all erasure errors of size d.