CSE 532
Assignment 4
Due Thursday, March 5, 1997

1. In this problem you will demonstrate the compactness theorem for propositional logic. A set of propositional formulas is satisfiable if there is a single assignment to all the variables that satisfies all the formulas simultaneously. Let Γ be countable set of propositional formulas in the propositional variables $x_1, x_2, x_3, \ldots$ Prove that if every finite subset of Γ is satisfiable then so is Γ, or put another way, if Γ is unsatisfiable then some finite subset of Γ is also. There is an easy proof of the compactness theorem given that there exists a complete and sound classical proof system. This is not what I want. You should prove the compactness theorem from first principles. (Hint, you should be able to generalize the proof of the the correctness of the resolution algorithm to prove this.)

2. Consider the system CTL. Show that $A(\alpha U \beta)$ is definable in terms of the operators $EG\gamma$ and $E(\gamma U \delta)$.

3. Again, consider the system CTL. Prove that CTL satisfiability is deterministic exponential time hard for log space reductions. That is, show that if $L \in \bigcup_c TIME(2^{cn})$ then L is reducible in log space to CTL-satisfiability. (Hint: this can be done using only the operators AG, AX, and EX.)

4. Consider the following axioms A in a single binary relation $\{<\}$.

$$\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$$
$$\forall x (\neg (x < x))$$
$$\forall x \forall y (x < y \vee y < x \vee x = y)$$
$$\forall x \forall y \exists z (x < y \rightarrow (x < z \wedge z < y))$$
$$\forall x \exists y (x < y)$$
$$\forall x \exists y (y < x)$$

Define $\Gamma = \mathrm{CN}(A)$.

(a) Prove that if $M = (\mathcal{U}^M, <^M)$ is countable and $M \models \Gamma$ then M is isomorphic to the rational numbers with the normal less than relation. If Q is the set of rationals and $<$ is the normal less than relation then an isomorphism is a one-to-one, onto function ϕ from $\mathcal{U}^M$ to Q with the property that for all $x, y \in \mathcal{U}^M$, $x <^M y$ if and only if $\phi(x) < \phi(y)$.

(b) Prove that Γ is a complete theory.

(c) Argue that Γ is decidable just using the fact that Γ is a complete theory with a finite set of axioms.

5. (Extra credit) Prove that LTL model checking is PSPACE hard for log space reductions. That is, given an LTL formula, a finite Kripke model and state in the model, show that the problem of determining if the formula is true for all paths starting at the state is PSPACE hard.

6. (Extra credit) Suppose we have two knowledge operators K_1 and K_2. So we can have formulas where these knowledge operators interact, for example, $K_1 K_2 \alpha$ states that agent 1 knows that agent 2 knows α. A model has two equivalence relations, one for K_1 and one for K_2. Show that satisfiablity for two knowledge operators is PSPACE hard for log space reductions.