CSE 532
Assignment 3
Due Thursday, February 19, 1997

1. In this problem we will explore a logic problem that is P-complete. A problem is P-complete if it is in P and every problem in P is log space reducible to it. A classic P-complete problem is the circuit value problem (cf. page 168 of Papadimitriou). It plays the roll in the study of P-completeness that CNF-SAT does for NP-completeness.

   A Horn clause is one of the form $(X_1 \wedge X_2 \wedge \cdots \wedge X_k) \rightarrow Y$ where $X_1, \ldots, X_k, Y$ are variables (not arbitrary literals). Prove that the problem of checking the satisfiability of a set of Horn clauses is P-complete. (Hint: you may want to use a reduction from the circuit value problem).

2. An interesting modal operator is the "knowledge" operator with the following interpretation. A *Kripke structure for knowledge* is a triple $M = (S, \pi, \sim)$ where S is a set of states, π maps S to truth assignments and $\sim$ is an equivalence relation. The knowledge operator K is a new unary operator. So if α is a formula then so is $K\alpha$. We define what it means for a formula to be true in a state s of M as follows.

$$\begin{aligned}
M, s &\models X \text{ if } \pi(s)(X) = T, \\
M, s &\models \alpha \vee \beta \text{ if } M, s \models \alpha \text{ or } M, s \models \beta, \\
M, s &\models \neg\alpha \text{ if } M, s \not\models \alpha, \\
M, s &\models K\alpha \text{ if for all } t \sim s, M, t \models \alpha.
\end{aligned}$$

   Additional Boolean operators are defined in the usual way. Validity and satisfiability are defined in the usual way. The system of knowledge formulas and their semantics is often called S_5.

   (a) Argue that the following are valid formulas for any α: $K\alpha \rightarrow \alpha$, $K\alpha \rightarrow KK\alpha$ and $\neg K\alpha \rightarrow K\neg K\alpha$.

   (b) Argue that the following are not valid for some α and β: $\alpha \rightarrow K\alpha$ and $K(\alpha \vee \beta) \rightarrow K\alpha \vee K\beta$.

   (c) Prove that satisfiability for knowledge formulas is NP-complete. (Hint: You can argue that every formula that is satisfiable is satisfied in a small Kripke structure.)

3. Consider Shoenfield's proof system:

   (a) $\frac{}{\neg\alpha\vee\alpha}$

   (b) $\frac{\alpha}{\beta\vee\alpha}$

   (c) $\frac{\alpha\vee\alpha}{\alpha}$

   (d) $\frac{(\alpha\vee\beta)\vee\gamma}{\alpha\vee(\beta\vee\gamma)}$

   (e) $\frac{\alpha\vee\beta,\neg\alpha\vee\gamma}{\beta\vee\gamma}$

   Assume that we have already proven that for all $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$, if $\vdash \alpha_{i_1} \vee \alpha_{i_2} \vee \cdots \vee \alpha_{i_k}$ then $\vdash \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$. So you never have to worry about the ordering or grouping of terms in a disjunction.

   Prove the deduction theorem that $\Gamma \cup \{\alpha\} \vdash \beta$ implies $\Gamma \vdash \neg\alpha \vee \beta$. (Hint: a natural approach is to prove it by induction on proof length.)

4. A first order theory is one that allow quantification over elements of a universe. One of the simplest first order theories is the theory of equality. In this case we consider all formulas that allow the binary relation $=$ that is always interpreted as equality. For example an equality formula might have the form $\forall x \exists y \exists z (x = y \wedge \neg(x = z))$. In this case a model is simply a non-empty set M. For a formula α we say $M \models \alpha$ if α is true in the model M. For example, the formula above is true in any model with at least two elements, but is not true in a model with exactly one element. Define Th($=$) to be the set of all valid equality formulas.

   (a) Prove that Th($=$) is PSPACE hard for log reductions. That is, every language in PSPACE is log space reducible to Th($=$). (Hint: it suffices to show that B_ω is log space reducible to Th($=$).)

   (b) (Extra Credit) Prove that Th($=$) is in PSPACE.